

Coalitional Bargaining Games with Random Proposers: Theory and Application*

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Abstract

We consider a non-cooperative coalitional bargaining game with random proposers to bridge a gap between non-cooperative game theory and cooperative game theory. Theoretical results include the existence of a stationary subgame perfect equilibrium (SSPE) and the characterization of the grand-coalition SSPE as a generalized Nash bargaining solution, provided that it lies in the core. We also prove that the grand-coalition SSPE is a unique symmetric SSPE in a symmetric game with nonempty core. In the last part, we apply the bargaining model to a production economy with one employer and multiple workers. If players are sufficiently patient, the economy has a unique SSPE allocation. The equilibrium allocation is compared with cooperative solutions such as the core, the Shapley value and the nucleolus. The SSPE allocation and the nucleolus have similar distributional properties.

Key words: non-cooperative coalitional bargaining, random proposers, Nash program, production economy, core, nucleolus

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1 Introduction

Game theory is traditionally divided into two branches, non-cooperative game theory and cooperative game theory. The two theories have been applied to the problem of multilateral bargaining in different perspectives. It is desirable to explore similarities and differences of the two theories for our better understanding of multilateral bargaining problems. Nash (1951) proposed a research program, now called the Nash program, to analyze cooperative games in the framework of non-cooperative game theory, by formulating preplay negotiations in cooperative games as moves in a larger non-cooperative game in extensive form. Since the seminal work of Rubinstein's (1982) two-person alternating-offers model, several extensions of it to an n -person game in coalitional form have been studied in the literature. Following the Nash program, this paper attempts to bridge a gap between two different branches in game theory.¹

To motivate our work, let us start to consider the classical problem of game theory, three-person symmetric games. The game situation is as follows. There are three players. If all three players cooperate, they earn a total payoff one. If any of two players cooperate, they earn a total payoff a ($0 \leq a \leq 1$). Any single player obtains zero payoff. The central question in the classic work of von Neumann and Morgenstern (1944) is: which coalition of players will form, and how will its members divide the coalitional payoff? Although a large volume of literature has studied this problem, there exists no agreement about a solution of it among researchers. In cooperative game theory, there have been proposed many solutions, and they have different predictions on coalition formation and payoff allocation. Some solutions such as the core, von-Neumann/Morgenstern's solution and Shapley value (Shapley, 1953) predict the efficient outcome of the three-player coalition. Other solutions such as the bargaining set (Aumann and Maschler 1964) and the nucleolus (Schmeidler,

¹Abbink et al. (2003) write that "for a long time, the two approaches seem to have found a form of peaceful coexistence with very little interaction."

1969) predict only a payoff allocation given a formation of coalitions. Although each of cooperative solutions is defined by its own interesting idea, it is not very clear how these solutions are related. Compared with cooperative solutions, it seems to us that the following prediction is more intuitive, at least not unreasonable. If the two-person coalition payoff a is very small (say, $a = 0.1$), all players cooperate and they divide the total payoff one equally. If parameter a is very large (say, $a = 0.9$), the three-person coalition is not attractive any more, and thus a two-person coalition forms. On the other hand, if parameter a is in a middle range, it is rather hard to say which coalition forms, a two-person coalition or the three-person coalition. In this case, we anticipate that there would be a positive probability for every coalition to be formed. A non-cooperative bargaining model with random proposers to be considered in this paper precisely predicts this outcome (see Example 4.1). The aim of this paper is to present some theoretical results of the bargaining model and to apply it to a production economy of one employer and multiple workers.

The random-proposer model has the following rule of negotiations. In the beginning of every bargaining round, one player is randomly selected as a proposer among “active” players who have not joined any coalitions in previous rounds. The probability for each player to be selected as a proposer is called her recognition probability. The selected player proposes a coalition of active players and a payoff allocation in the coalition. Thereafter, all other members in the coalition either accept or reject the proposal sequentially. The proposal is agreed by unanimity. If it is agreed, then the members of the coalition quit the game and other players continue negotiations in the next round under the same rule. If the proposal is rejected, then the same set of players repeat a bargaining round. Negotiations stop if all players belong to any coalitions. Players discount future payoffs.

The results of the paper are as follows. In the first part, we prove the existence of a stationary subgame perfect equilibrium (SSPE) in a general situation where the probability distribution of players’ recognition is arbitrary

and players may have different discount factors for future payoffs. The no-delay of an agreement is proved. We next consider an efficient SSPE in which the grand coalition forms with probability one. It is shown that such an efficient SSPE (called the *grand-coalition SSPE*) exists if and only if the expected payoffs for players belong to an enlarged set of the core (called an ε -core in the literature). This enlarged core shrinks to the usual core in the limit that players' discount factors go to one. The expected payoff of every player i in the grand coalition is proportional to the ratio $p_i/(1 - \delta_i)$ where p_i is player i 's recognition probability, and δ_i is her discount factor. The ratio represents the bargaining power of the player. In the case that all players have a common discount factor, an equilibrium agreement converges to a generalized Nash bargaining solution with player i 's weight p_i , in the limit that players are sufficiently patient. The uniqueness of an SSPE payoff for a general coalitional game is an open question in the literature. We prove that the grand-coalition SSPE is a unique symmetric SSPE in a symmetric game with nonempty core for any discount factor for future payoffs.

In the last part, the bargaining model is applied to a production economy with one employer and multiple workers. All players have the same discount factors for future payoffs, and the recognition probability is uniform. The economy has a non-empty core. In the limit that the discount factor is almost equal to one, we prove that the economy has a unique SSPE payoff allocation and it is efficient. The equal allocation is attained if and only if full-employment has the highest productivity per capita. This condition is equivalent to that the equal allocation is in the core. If the equal allocation does not belong to the core, the SSPE allocation is on the boundary of the core, where the employer receives the smallest payoff in the core. Comparing the SSPE allocation with Shapley value and the nucleolus, we show that the SSPE allocation and the nucleolus have similar distributional properties.

The literature on the random-proposer model for coalitional bargaining is growing. We here review only those which are closely related to the present

paper. The random-proposer model has been extensively studied for legislative bargaining since a seminal paper of Baron and Ferejohn (1989). They characterized a unique SSPE payoff for a majority voting game when voters are identical in recognition probability and discount factors for future payoffs. Eraslan (2002) proved the existence of an SSPE and the uniqueness of an SSPE payoff in a general case that voters may be different in recognition probability and discount factors. Eraslan and McLennan (2006) extended the uniqueness result to a general voting game. Norman (2002) studied the finite-horizon version of the Baron-Ferejohn model. Montero (2002) considered a special class of weighted majority voting games (called apex games) and characterized the kernel as a unique SSPE payoff under two (egalitarian and proportional) recognition rules. Montero (2006) characterized the nucleolus as a self-confirming power index for a majority voting game in the case that voters have the same discount factors.

The random-proposer model was applied to an n -person super-additive coalitional game with transferable utility in Okada (1996). In the case that players are identical in recognition probability and discount factors, we showed no-delay of agreement in an SSPE, and derived a necessary and sufficient condition for the grand-coalition SSPE to exist. The efficiency result has been generalized in several directions. Okada (2005) extended the result to an n -person cooperative game in strategic form, and characterized the grand-coalition SSPE payoff as a generalized Nash bargaining solution in the case of a common discount factor. Yan (2002, 2005) studied the model in a restricted case that the game stops after one coalition forms (one-stage property). In the case that players have the common discount factors, Yan (2002) proved that every core allocation can be sustained as a unique SSPE payoff if (after normalization) it is employed as the recognition probability. Yan (2005) proved the uniqueness of an SSPE payoff for a symmetric game with one-stage property. Recently, Compte and Jehiel (2008) consider an asymptotic efficiency of an SSPE where the probability of a subcoalition converges to zero as players be-

come sufficiently patient. In games with one-stage property, they characterize an asymptotic efficient equilibrium allocation as the core allocation (called the coalitional Nash bargaining solution) which maximizes the Nash product over the core. Finally, Montero and Okada (2007) show non-uniqueness of SSPE payoffs in a three-person game with discrete feasible payoffs.

The paper is organized as follows. Section 2 presents the random-proposer model for an n -person coalitional game with transferable utility. Section 3 proves the existence of a behavior-strategy SSPE. Section 4 characterizes the grand-coalition SSPE. Section 5 analyzes a production economy with one employer and multiple workers. Proofs are given in the Appendix. Section 6 concludes the paper.

2 Definitions

We consider an n -person game (N, v) in coalitional form with transferable utility. $N = \{1, 2, \dots, n\}$ is the set of players. A nonempty subset S of N is called a *coalition* of players. Let $\mathcal{C}(N)$ be the set of all coalitions of N . The *characteristic function* v is a real-valued function on $\mathcal{C}(N)$ satisfying (1) (zero-normalized) $v(\{i\}) = 0$ for all $i \in N$, (2) (super-additive) $v(S \cup T) \geq v(S) + v(T)$ for any two disjoint coalitions S and T , and (3) (essential) $v(N) > 0$. For coalition S , $v(S)$ is interpreted to be a sum of money that the members of S can distribute in any way if they agree to it.

A payoff allocation for coalition S is a vector $x^S = (x_i^S)_{i \in S}$ of real numbers where x_i^S represents a payoff to player $i \in S$. A payoff allocation x^S for S is *feasible* if $\sum_{i \in S} x_i^S \leq v(S)$. Let X^S denote the set of all feasible payoff allocations for S , and let X_+^S denote the set of all elements in X^S with non-negative components. For a finite set T , the notation $\Delta(T)$ denotes the set of all probability distributions on T

As a non-cooperative bargaining procedure for a game (N, v) , we consider the random proposer model presented in Okada (1996). Let p be a function

which assigns to every coalition $A \subset N$ a probability distribution $p^A \in \Delta(A)$. The interpretation of p is that the distribution p^A selects a proposer $i \in A$ randomly when A is the set of all active players in negotiations. Following the literature on legislative bargaining, we will call p the *recognition probability*.

The bargaining model has the following rule. Negotiations take place over a (possibly) infinite number of bargaining rounds t ($= 1, 2, \dots$). Let $N_t(\subset N)$ be the set of all “active” players who do not belong to any coalitions in round t . In the initial round, we put $N_1 = N$. The bargaining process in round t runs as follows. In the beginning, a player $i \in N_t$ is randomly selected as a proposer according to the probability distribution $p^{N_t} \in \Delta(N_t)$. The selected player i proposes a coalition S with $i \in S \subset N_t$ and a payoff allocation $x^S \in X_+^S$. Then, all other members in S either accept or reject the proposal (S, x^S) sequentially. The order of responders does not affect the result in any critical way. If all responders accept the proposal, then coalition S forms, and the members of S quit the game, receiving payoff x^S . Thereafter, negotiations go to the next round $t+1$ with $N^{t+1} = N^t - S$. If any one responder rejects the proposal, then negotiations continue in the next round $t+1$ with $N^{t+1} = N^t$. The bargaining process ends when every player in N joins some coalition.

The payoffs of players are defined as follows. When a proposal (S, x^S) is agreed in round t , the payoff of every player $i \in S$ is $\delta_i^{t-1} x_i^S$ where δ_i ($0 \leq \delta_i < 1$) is player i 's discount factor for future payoffs. When the bargaining does not stop, all players who fail to join any coalitions receive zero payoffs. In the model, all players have perfect information about game play when they make decisions.

The bargaining model above is denoted by $\Gamma(N, p, \delta)$ where N is the initial set of players, p the random rule of selecting proposers, and $\delta = (\delta_1, \dots, \delta_n)$ players' discount factors for future payoffs. Formally, the bargaining model $\Gamma(N, p, \delta)$ is represented as an infinite-length extensive game with perfect information and with chance moves. The rule of the game is the common knowledge of players.

A (*behavior*) *strategy* for player i in $\Gamma(N, p, \delta)$ is defined in a standard manner. A history h_i^t before player i 's move in round t is a sequence of all past actions in $\Gamma(N, p, \delta)$ including the selections of proposers. Roughly, a strategy σ_i of player i is a function which assigns her (randomized) action $\sigma_i(h_i^t)$ to every possible history h_i^t . Specifically, when player i is a proposer in round t , $\sigma_i(h_i^t)$ is a probability distribution (with a finite support) on the set of all possible proposals (S, x^S) with $i \in S \subset N^t$ and $x^S \in X_+^S$. When player i is a responder in round t , $\sigma_i(h_i^t)$ is in $\Delta(\{accept, reject\})$. For a strategy combination $\sigma = (\sigma_1, \dots, \sigma_n)$, the expected (discounted) payoff for player i in $\Gamma(N, p, \delta)$ can be defined in a usual way.

For every coalition $S \in \mathcal{C}(N)$, a subgame of the extensive game $\Gamma(N, p, \delta)$ which starts after the coalition S has formed is identical to the bargaining model $\Gamma(N - S, p, \delta)$. A strategy σ_i for every player i naturally induces her strategy in the subgame $\Gamma(N - S, p, \delta)$. A strategy σ_i for player i in $\Gamma(N, p, \delta)$ is called *stationary* if player i 's action depends only on payoff-relevant history, not on a whole part of the history. A payoff-relevant history for player i consists of the set N_t of active players in negotiations when she is a proposer, and it also includes a proposal made in the present period when she is a responder.

The solution concept that we apply to the bargaining model $\Gamma(N, p, \delta)$ is a stationary subgame perfect equilibrium.

Definition 2.1. A strategy combination $\sigma^* = (\sigma_1^*, \dots, \sigma_n^*)$ of $\Gamma(N, p, \delta)$ is called a *stationary subgame perfect equilibrium (SSPE)* if σ^* is a subgame perfect equilibrium of $\Gamma(N, p, \delta)$ and every player i 's strategy σ_i^* is stationary.

It is well-known that in a broad class of Rubinstein-type sequential multi-lateral bargaining games including our model $\Gamma(N, p, \delta)$, there is a large multiplicity of (non-stationary) subgame perfect equilibria when the discount factor of future payoffs is sufficiently close to one (see Sutton 1986 and Osborne and Rubinstein 1990). The multiplicity of subgame perfect equilibria holds even

in the n -person pure bargaining game where no subcoalitions are allowed. An SSPE is the simplest type of a subgame perfect equilibrium and thus it may be easier for players to coordinate their mutual expectations on it (see Baron and Kalai 1993). The SSPE is a natural reference point of the analysis in multilateral bargaining models.

3 Equilibrium configuration

For an SSPE $\sigma = (\sigma_1, \dots, \sigma_n)$ of the game $\Gamma(N, p, \delta)$ and a coalition S , let v_i^S be the expected payoff of player i in the subgame $\Gamma(S, p, \delta)$, and let q_i^S be the probability distribution according to which player i chooses coalitions T , $i \in T \subset S$, in $\Gamma(S, p, \delta)$. Let $v^S = (v_i^S)_{i \in S}$ and $q^S = (q_i^S)_{i \in S}$. We call the collection $(v^S, q^S)_{S \in \mathcal{C}(N)}$ the *configuration* of an SSPE σ .

Definition 3.1. A collection $(v^S, q^S)_{S \in \mathcal{C}(N)}$ of players' expected payoffs v^S and their randomized choices q^S of coalitions for all sets S of active players is called an *equilibrium configuration* for $\Gamma(N, p, \delta)$ if it is a configuration of some SSPE σ for $\Gamma(N, p, \delta)$.

The next lemma plays a critical role in characterizing an SSPE. The lemma was first proved in Okada (1996) in the case that all players are identical in recognition probability and discount factors.

Lemma 3.1. In every SSPE $\sigma = (\sigma_1, \dots, \sigma_n)$ for $\Gamma(N, p, \delta)$, every player's proposal is accepted in the initial round of negotiations. In the proposal, all other members j in the coalition are offered their discounted expected payoffs $\delta_j v_j^N$.

Proof. For every $i \in N$, let v_i be player i 's expected payoff for σ in $\Gamma(N, p, \delta)$.

By the rule of $\Gamma(N, p, \delta)$, the super-additivity of v yields

$$\sum_{i \in N} v_i \leq v(N) \quad \text{and} \quad v_i \geq 0 \quad \text{for all } i \in N.$$

Consider the maximization problem

$$\begin{aligned} \max_{S, y} \quad & v(S) - \sum_{j \in S, j \neq i} y_j & (1) \\ \text{s.t.} \quad & \text{(i) } i \in S \subset N, \quad y \in X_+^S \\ & \text{(ii) } y_j \geq \delta_j v_j \quad \text{for all } j \in S, j \neq i. \end{aligned}$$

Let m^i be the optimal value of (1). Since $(N, (\delta_1 v_1, \dots, \delta_n v_n))$ is a feasible solution, we have

$$m^i \geq v(N) - \sum_{j \in N, j \neq i} \delta_j v_j \geq \delta_i v_i. \quad (2)$$

If $m^i = \delta_i v_i$, then (2) implies

$$v(N) = \sum_{j \in N} \delta_j v_j. \quad (3)$$

Since

$$v(N) \geq \sum_{j \in N} v_j \geq \sum_{j \in N} \delta_j v_j,$$

(3) implies $\sum_{j \in N} v_j = \sum_{j \in N} \delta_j v_j$. Since $\delta_j < 1$ for all j , we must have $v_j = 0$ for all $j \in N$. This yields $v(N) = 0$ from (3), which contradicts $v(N) > 0$.

Therefore, we obtain $m^i > \delta_i v_i$.

Let (S, y^S) be the optimal solution of (1). Then, (S, y^S) must satisfy

$$\begin{aligned} m^i &= v(S) - \sum_{j \in S, j \neq i} y_j^S \\ y_j^S &= \delta_j v_j, \quad \forall j \in S, j \neq i. \end{aligned}$$

For a sufficiently small $\epsilon > 0$, define a payoff allocation $z^S \in X_+^S$ such that

$$z_i^S = m^i - \epsilon, \quad z_j^S = \delta_j v_j + \frac{\epsilon}{s-1}, \quad \forall j \in S, j \neq i,$$

where s is the number of all members in S . If player i proposes (S, z^S) , then this is accepted in σ since all responders j receive only the discounted payoffs $\delta_j v_j$ if they reject the proposal. Since $z_i^S = m^i - \epsilon > \delta_i v_i$ for any sufficiently small $\epsilon > 0$, the conditional expected payoff for player i in σ when she is a proposer is strictly greater than $\delta_i v_i$, which is her continuation value when her proposal is rejected. This fact implies that player i 's proposal must be accepted in σ . QED.

An important implication of this lemma is that an SSPE σ of the bargaining model $\Gamma(N, p, \delta)$ is determined uniquely (up to responses off equilibrium path) by its configuration $(v^S, q^S)_{S \in \mathcal{C}(N)}$. When player i chooses a coalition S in σ in the first round, she proposes the payoff allocation $x^S = (x_j^S)_{j \in S}$ for coalition S such that

$$x_i^S = v(S) - \sum_{j \in S, j \neq i} \delta_j v_j^N, \quad x_j^S = \delta_j v_j^N, \quad \forall j \in S, j \neq i,$$

and all other members of S accept it. Another implication of the lemma is that an agreement of a coalition (possibly inefficient) is made in the first round in the bargaining game $\Gamma(N, p, \delta)$, regardless of a proposer, if the characteristic function is super-additive. That is, the bargaining game has no delay of agreements. This result does not hold in other bargaining games where the first rejector becomes the next proposer (see Chatterjee, Dutta, Ray and Sengupta 1993, and Ray and Vohra 1999).

The next step of our analysis is to characterize an equilibrium configuration in $\Gamma(N, p, \delta)$.

Lemma 3.2. In a subgame $\Gamma(S, p, \delta)$, let p_i^S be player i 's recognition probability, v_i^S player i 's expected payoff, and q_i^S player i 's randomized choice of coalitions where a coalition $T \subset S$ is chosen with probability $q_i^S(T)$. A collection $(v^S, q^S)_{S \in \mathcal{C}(N)}$, $v^S = (v_i^S)_{i \in N}$ and $q^S = (q_i^S)_{i \in N}$, is an equilibrium configuration of $\Gamma(N, p, \delta)$ if and only if the following conditions hold for every coalition $S \in \mathcal{C}(N)$ and every $i \in S$:

(i) $q_i^S(\hat{S}) > 0$ implies that \hat{S} is a solution of

$$\max_{i \in T \subset S} (v(T) - \sum_{j \in T, j \neq i} \delta_j v_j^S), \quad (4)$$

(ii) $v_i^S \in R_+$ satisfies

$$\begin{aligned} v_i^S &= p_i^S \max_{i \in T \subset S} (v(T) - \sum_{j \in T, j \neq i} \delta_j v_j^S) \\ &+ \sum_{j \in S, j \neq i} p_j^S \delta_i \left(\sum_{j \in T \subset S, i \in T} q_j^S(T) v_i^S + \sum_{j \in T \subset S, i \notin T} q_j^S(T) v_i^{S-T} \right). \end{aligned} \quad (5)$$

Proof. It follows from Lemma 3.1 that, in an SSPE σ of $\Gamma(N, p, \delta)$, player i receives the payoff defined by (4) when she is a proposer, and receives $\delta_i v_i^S$ and $\delta_i v_i^{S-T}$ as a responder, when player $j (\neq i)$ proposes coalitions T including i and T excluding i , respectively. Noting this fact, we can see that condition (i) means that player i 's randomized choice q_i^S of coalitions composes her locally optimal choice when she is selected as a proposer, and that condition (ii) defines the expected payoffs $(v_i^S)_{S \in \mathcal{C}(N)}$ of player i recursively. The if-part can be proved by a well-known fact (sometimes called the single-period deviation property) that the local optimality of a strategy implies the global optimality in an infinite-length extensive game with perfect information such as the bargaining game $\Gamma(N, p, \delta)$ (see Selten 1981, p.137). QED

In Lemma 3.2, we call condition (i) the *optimality condition* and condition (ii) the *payoff equation* for an equilibrium configuration in $\Gamma(N, p, \delta)$. With

help of these two conditions, we will characterize an SSPE of the bargaining model $\Gamma(N, p, \delta)$ in the next section.

With help of Lemma 3.2, the existence of an SSPE in the bargaining model $\Gamma(N, p, \delta)$ can be proved by Kakutani's fixed point theorem in a standard manner. The existence of an SSPE in related models is proved in the literature (Ray and Vohra 1999, Eraslan 2002, Gomes 2005 among others).

Proposition 3.1. There exists an SSPE of the bargaining model $\Gamma(N, p, \delta)$.

Proof. By Lemma 3.2, it suffices to prove that there exists a collection $(v^S, q^S)_{S \in \mathcal{C}(N)}$ of players' expected payoffs $v^S = (v_i^S)_{i \in S}$ and their randomized choices $q^S = (q_i^S)_{i \in S}$ of coalitions in all subgames $\Gamma(S, p, \delta)$ such that (4) and (5) hold for every coalition $S \in \mathcal{C}(N)$ and every $i \in S$. We prove this claim by induction regarding the number s of players in coalition S . When $s = 1$ where $S = \{i\}$, the claim trivially holds by putting $v_i^{\{i\}} = 0$ and $q_i^{\{i\}}(\{i\}) = 1$. For any $2 \leq s \leq n$, suppose that the claim holds for all $t = 1, \dots, s - 1$. Let $S \in \mathcal{C}(N)$ be any coalition with s members. For all proper subsets T of S , let $v^T = (v_j^T)_{j \in T}$ be expected payoffs for members in T and let $q^T = (q_i^T)_{i \in T}$ be their randomized choices of coalitions in the subgame $\Gamma(T, p, \delta)$ such that (4) and (5) hold. The existence of such v^T and q^T is guaranteed by the supposition of induction.

For every $i \in S$, let $\Delta_i^S = \Delta(\{T \mid i \in T \subset S\})$. Recall that $\Delta(A)$ is the set of all probability distributions on a finite set A . Δ_i^S is the set of player i 's all randomized choices of coalitions when she is selected as a proposer. Define a multi-valued mapping F from a compact and convex set $X_+^S \times \prod_{i \in S} \Delta_i^S$ to itself as follows. For $(x, q) \in X_+^S \times \prod_{i \in S} \Delta_i^S$, $F(x, q)$ is the set of all $(y, r) \in X_+^S \times \prod_{i \in S} \Delta_i^S$ which satisfy for all $i \in S$

(i) $r_i \in \Delta(\{\hat{S} \mid i \in \hat{S} \subset S \text{ and } \hat{S} \text{ is a solution of}$

$$\max_{i \in T \subset S} (v(T) - \sum_{j \in T, j \neq i} \delta_j x_j)\},$$

and

(ii) $y_i \in R_+$ satisfies

$$\begin{aligned} y_i &= p_i^S \max_{i \in T \subset S} (v(T) - \sum_{j \in T, j \neq i} \delta_j x_j) \\ &\quad + \sum_{j \in S, j \neq i} p_j^S \delta_i \left(\sum_{j \in T \subset S, i \in T} r_j(T) x_i + \sum_{j \in T \subset S, i \notin T} r_j(T) v_i^{S-T} \right), \end{aligned}$$

where $r_j(T)$ is the probability which r_j assigns to $T \subset S$.

We can show without much difficulty that $F(x, q)$ is a non-empty convex set in $X_+^S \times \prod_{i \in S} \Delta_i^S$, and that F is upper-semicontinuous. By Kakutani's fixed point theorem, there exists a fixed point (x^*, q^*) of F with $(x^*, q^*) \in F(x^*, q^*)$. Now, we define $v_i^S = x_i^*$ and $q_i^S = q_i^*$ for all $i \in S$. Then, the fact that $(v^S, q^S) \in F(v^S, q^S)$ implies that $v^S = (v_i^S)_{i \in S}$ and $q^S = (q_i^S)_{i \in S}$, together with (v^T, q^T) for all proper subsets T of S , satisfy (4) and (5) in Lemma 3.2 for every $i \in S$. QED

We remark that Proposition 3.1 does not hold for a pure-strategy SSPE. To obtain the existence of an SSPE, one needs to allow a randomized choice of coalitions by proposers. As we will show in the next section, a pure-strategy SSPE with an agreement of the grand coalition N exists if the expected equilibrium payoff allocation is in the (enlarged) core of the underlying game.

4 Characterizations

Let $p_i = p_i^N$ for every $i \in N$. The payoff equation (5) for an equilibrium configuration $(v^S, q^S)_{S \in \mathcal{C}(N)}$ in $\Gamma(N, p, \delta)$ can be rewritten as

$$\begin{aligned} & (1 - \delta_i \sum_{j \in N, j \neq i} p_j \sum_{i, j \in T \subset N} q_j^N(T)) v_i^N + p_i \sum_{j \in N, j \neq i} \left(\sum_{i, j \in S \subset N} q_i^N(S) \right) \delta_j v_j^N \\ &= p_i \sum_{i \in S \subset N} q_i^N(S) v(S) + \delta_i \sum_{j \in N, j \neq i} p_j \sum_{j \in T \subset N, i \notin T} q_j^N(T) v_i^{N-T}. \end{aligned} \quad (6)$$

We first consider under what conditions the grand coalition N is formed, independent of a proposer.

Definition 4.1. A behavior strategy σ for $\Gamma(N, p, \delta)$ is called the *grand-coalition SSPE* if it is an SSPE of $\Gamma(N, p, \delta)$ and the grand coalition N forms, independent of a proposer.

The next theorem characterizes the grand-coalition SSPE.

Theorem 4.1. The grand-coalition SSPE of $\Gamma(N, p, \delta)$ is characterized as follows.

(i) The expected payoff v_i for player i is given by

$$v_i = \frac{\frac{p_i}{1-\delta_i}}{\sum_{j \in N} \frac{p_j}{1-\delta_j}} v(N). \quad (7)$$

The grand-coalition SSPE exists if and only if

$$\sum_{i \in S} v_i + \sum_{j \in N-S} v_j(1-\delta_j) \geq v(S) \text{ for all } S \subset N. \quad (8)$$

(ii) If all players have the common discount factors δ , then the expected payoff v_i for player i is given by $p_i v(N)$. For any δ almost close to one, the grand-coalition SSPE exists if and only if its expected payoff vector $(p_1 v(N), \dots, p_n v(N))$ is in the core of (N, v) . Every player proposes the payoff vector $(p_1 v(N), \dots, p_n v(N))$ in the limit that δ goes to one.

Proof. (i) The payoff equation of the grand-coalition SSPE is given by

$$\begin{pmatrix} 1 - \delta_1(1 - p_1) & \delta_2 p_1 & \dots & \delta_n p_1 \\ \delta_1 p_2 & 1 - \delta_2(1 - p_2) & \dots & \delta_n p_2 \\ \vdots & \vdots & \ddots & \vdots \\ \delta_1 p_n & \delta_2 p_n & \dots & 1 - \delta_n(1 - p_n) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} p_1 v(N) \\ p_2 v(N) \\ \vdots \\ p_n v(N) \end{pmatrix}.$$

Calculating the i -th row in the equation above, we obtain

$$(1 - \delta_i)v_i + p_i \sum_{j \in N} \delta_j v_j = p_i v(N), \quad \forall i \in N. \quad (9)$$

Summing up both sides of (9) for all $i \in N$ yields

$$\sum_{i \in N} v_i = v(N). \quad (10)$$

Letting $k = \sum_{i \in N} \delta_i v_i$, (9) is rewritten as

$$v_i = \frac{p_i}{1 - \delta_i} (v(N) - k), \quad \forall i \in N. \quad (11)$$

It follows from (11) that

$$k = \sum_{i \in N} \delta_i v_i = (v(N) - k) \sum_{i \in N} \frac{\delta_i}{1 - \delta_i} p_i.$$

This yields

$$k = \frac{\sum_{j \in N} \frac{\delta_j}{1 - \delta_j} p_j}{1 + \sum_{j \in N} \frac{\delta_j}{1 - \delta_j} p_j} v(N) = \frac{\sum_{j \in N} \frac{\delta_j}{1 - \delta_j} p_j}{\sum_{j \in N} \frac{1}{1 - \delta_j} p_j} v(N). \quad (12)$$

Substituting (12) into (11), we prove (7). The optimality condition (4) is given by

$$v(N) - \sum_{j \in N, j \neq i} \delta_j v_j \geq v(S) - \sum_{j \in S, j \neq i} \delta_j v_j, \quad \forall S \subset N.$$

In view of (10), we can see that the optimality condition is equivalent to (8).

By Lemma 3.2, we can prove the last part of (1).

(ii) When $\delta_1 = \dots = \delta_n = \delta$, it can be easily shown that (7) implies $v_i = p_i v(N)$. Note that v_i is independent of δ . Suppose first that there exists the grand-coalition SSPE for any δ almost close to one. Taking $\delta \rightarrow 1$ in (8), we obtain $\sum_{i \in S} v_i \geq v(S)$ for all $S \subset N$. This means that the expected payoff vector $(p_1 v(N), \dots, p_n v(N))$ is in the core of (N, v) . Conversely, suppose that

$\sum_{i \in S} v_i = \sum_{i \in S} p_i v(N) \geq v(S)$ for all $S \subset N$. Since $v_j > 0$ for all $j \in N - S$, we can show that (8) holds for any $\delta < 1$. By Lemma 3.2, this implies that there exists the grand-coalition SSPE of $\Gamma(N, p, \delta)$ for any δ , and it has the expected payoff vector $(p_1 v(N), \dots, p_n v(N))$. In the grand-coalition SSPE, player i proposes a payoff allocation $(\delta p_1 v(N), \dots, (1 - \delta \sum_{j \neq i} p_j) v(N), \dots, \delta p_n v(N))$. This proposal converges to $(p_1 v(N), \dots, p_n v(N))$ as $\delta \rightarrow 1$. QED

The theorem has several implications. First, it shows that when players form the grand coalition in the bargaining model $\Gamma(N, p, \delta)$, their expected payoffs are proportional to the ratio $\frac{p_i}{1 - \delta_i}$. This fact suggests that the ratio represents the bargaining power of each player. Players who have higher recognition probability and are more patient can receive higher expected payoffs. The random-proposer model explains the bargaining power of a player by two factors, procedural rules and players' time preferences.²

Second, the theorem shows that the existence of the grand-coalition SSPE for $\Gamma(N, p, \delta)$ is closely related to the non-emptiness of the core of the cooperative game (N, v) . (8) requires that the expected payoffs v_i for players satisfy, for all $S \subset N$,

$$\sum_{i \in S} v_i \geq v(S) - \epsilon, \quad \text{where } \epsilon = \sum_{j \in N - S} v_j (1 - \delta_j).$$

This means that the expected payoffs for players in the grand-coalition SSPE belongs to a larger set of the core of (N, v) , which is called the ϵ -core in the literature. As δ goes to one, this set converges to the core of (N, v) . Namely, the grand-coalition SSPE exists for any δ almost close to one if and only if its expected payoff vector is in the core of the game (N, v) . An immediate corollary of this result is that if the core of the game is empty, then the grand-coalition SSPE never exists, whatever recognition probability distribution is,

²One may interpret this result negatively, arguing that the equilibrium outcome of a non-cooperative bargaining model is sensitive to unimportant procedural details. In our view, the recognition probability for players should not be regarded as unimportant details.

when players are sufficiently patient.

Thirdly, when players have the common discount factors almost close to one, the proposal of every player in the grand-coalition SSPE is equal to a generalized Nash bargaining solution with weights (p_1, \dots, p_n) , which is a solution of

$$\max_x \prod_{i \in N} x_i^{p_i} \quad \text{s.t.} \quad \sum_{i \in N} x_i = v(N), \quad x_i \geq 0, \forall i \in N$$

where the disagreement point is given by players' payoffs $v(\{i\}) = 0$, $i = 1, \dots, n$ in the case of no coalitions.³

We remark that the grand-coalition SSPE and an inefficient SSPE may exist simultaneously. Yan (2000) proves that the grand-coalition SSPE is a unique SSPE in a game with one-stage property when the game has a nonempty core. The uniqueness of an SSPE for a general cooperative game is an open question. To restrict our analysis to a symmetric SSPE, we prove the uniqueness of the grand-coalition SSPE in an n -person symmetric game.

Definition 4.2.

- (i) A game (N, v) is called *symmetric* if the value $v(S)$ of coalition S depends only on the size s of S . Whenever no confusion arises, $v(S)$ is denoted by $v(s)$.
- (ii) Let (N, v) be a symmetric game. An SSPE σ of $\Gamma(N, p, \delta)$ is called *symmetric* if the following conditions hold: if any player i chooses a coalition S with positive probability in any round t , then every player $j \in N$ also proposes all coalitions (including herself) of the same size as S with (possibly unequal) positive probability in the same round.

A symmetric SSPE requires that all proposers' choices of coalitions are invariant to renaming of players. That is, all proposers treat equally all other

³Note that, if players have different discount factors, the expected payoff in (7) does not have a (unique) limit as players' discount factors go to one.

players as their coalition partners. A symmetric SSPE is a natural criterion of equilibrium selection in a symmetric game. It includes the grand-coalition SSPE and the Baron-Ferejohn equilibrium of a majority voting game in which every proposer chooses equally all minimal winning coalitions (including herself). In a general setup where coalitions form sequentially, the Baron-Ferejohn equilibrium can be generalized to an equilibrium in which all proposers choose randomly all coalitions of the same size in each round until all players join any coalition. Remark that a symmetric SSPE does not rule out the possibility that a proposer chooses randomly coalitions of different sizes in the same round.

Lemma 4.1. Let (N, v) be a symmetric game. In every symmetric SSPE of $\Gamma(N, p, \delta)$ except the grand-coalition SSPE, all players $i \in N$ receive the same discounted expected payoffs $\delta_i v_i$.

Proof. Let σ be a symmetric SSPE of $\Gamma(N, p, \delta)$ other than the grand-coalition SSPE. In σ , there exists some player i who proposes a subcoalition $S \neq N$ with positive probability. By the optimality condition of σ , it must hold that

$$v(S) - \sum_{j \in S, j \neq i} \delta_j v_j \geq v(T) - \sum_{j \in T, j \neq i} \delta_j v_j, \quad \forall T \subseteq N.$$

Pick any $j \in S, j \neq i$, and any $k \notin S$. Substituting $T = (S - \{j\}) \cup \{k\}$ into the condition above yields $\delta_k v_k \geq \delta_j v_j$. By the definition of a symmetric SSPE, player i proposes the coalition $(S - \{j\}) \cup \{k\}$ with positive probability. Replacing S with $(S - \{j\}) \cup \{k\}$ in the arguments above, we can obtain $\delta_j v_j \geq \delta_k v_k$. Thus, $\delta_j v_j = \delta_k v_k$. Finally, since σ is symmetric, player j also chooses S with positive probability. By repeating the same arguments as above, we can show that $\delta_i v_i = \delta_k v_k$. Thus, $\delta_i v_i = \delta_j v_j = \delta_k v_k$. QED

The lemma shows that when all players have a risk not to be invited to

coalitions in a symmetric game, their discounted expected payoffs are identical in equilibrium. This result is caused by a competition among players in coalition formation. Every player accepts to join any coalition if she is offered her discounted expected payoff. In this sense, the discounted expected payoff (called a continuation payoff in the bargaining literature) can be interpreted as a “price” which all proposers should pay when they want her as their coalitional partners. When a subcoalition is formed, any “expensive” player is excluded from the coalition. A mechanism similar to price competition makes all players’ prices equal.

Lemma 4.2. For an SSPE σ of $\Gamma(N, p, \delta)$ with a configuration $(v^S, q^S)_{S \in \mathcal{C}(N)}$, let v_i be the expected equilibrium payoff for player i , and let $V_\sigma = \sum_{i \in N} v_i$. Then,

$$V_\sigma = \sum_{i \in N} p_i \sum_{S: i \in S \in \mathcal{C}(N)} q_i^N(S) v(S) + \sum_{i \in N} \delta_i \alpha_i \quad (13)$$

where $\alpha_i = \sum_{j \neq i} p_j \sum_{i \notin T, j \in T} q_j^N(T) v_i^{N-T}$.

Proof. Summing up both sides of the payoff equation (6) for all $i \in N$, we obtain (13). QED

The RHS of (13) represents the discounted sum of coalitional values realized in equilibrium. In a majority voting game, V_σ is simply equal to the value of a winning coalition for all σ .

Proposition 4.2. Let (N, v) be a symmetric game, and let σ be a symmetric SSPE other than the grand-coalition SSPE. Then, the expected payoff v_i for player i in σ is given by

$$v_i = \frac{V_\sigma}{\delta_i \sum_{j \in N} \frac{1}{\delta_j}} \quad (14)$$

where V_σ is the discounted sum of coalitional values in σ defined by (13). An

SSPE σ exists only if

$$v(s_1) - v(t) \geq \frac{V_\sigma}{\sum_{j \in N} \frac{1}{\delta_j}} (s_1 - t) \text{ for all } t = 1, \dots, n. \quad (15)$$

where s_1 is the size of coalitions formed in the initial round.⁴

Proof. (14) follows from Lemma 4.1 and $V_\sigma = \sum_{i \in N} v_i$. The optimality condition for σ implies

$$v(s_1) - \sum_{j \in S_1, j \neq i} \delta_j v_j \geq v(T) - \sum_{k \in T, k \neq i} \delta_k v_k, \quad \forall T \subseteq N$$

where S_1 is a coalition formed in the initial round. With (14), this yields (15). QED

The proposition shows that the expected payoffs for players are inversely proportional to their discount factors in any symmetric SSPE other than the grand-coalition SSPE.⁵ This result is in stark contrast to Theorem 4.1. The conflict of coalition formation changes drastically the payoff distribution among players. Players with high discount factors receive less expected payoffs. Any “expensive” player who is patient is unlikely invited by other players to coalitions, and thus the player’s expected payoff decreases.

In a majority game (except the unanimous game), every proposer chooses a minimal winning coalition in every SSPE since the continuation payoffs (acceptance levels) of voters are strictly positive (Lemma 3.1). That is, the majority game does not have the grand-coalition SSPE. Proposition 4.2 implies that a symmetric SSPE payoff of a majority voting game is unique for any discount factor.

⁴The conditions similar to (15) also hold in all other rounds if subcoalitions form in these rounds. Since the proof is the same, we omit it.

⁵Kawamori (2005) derives (14) in a majority voting game in a restricted case that voters are sufficiently similar in recognition probability and discount factors.

Theorem 4.2. In a symmetric game (N, v) with non-empty core, the grand-coalition SSPE is a unique symmetric SSPE of $\Gamma(N, p, \delta)$ for any discount factors $\delta = (\delta_1, \dots, \delta_n)$.

Proof. Suppose that there exists a symmetric SSPE σ of $\Gamma(N, p, \delta)$ other than the grand-coalition SSPE. Let $s(< n)$ be the size of a coalition which may form with positive probability in the initial round in σ . Then, it follows from (15) that

$$v(s) - v(t) \geq \frac{V_\sigma}{\sum_{j \in N} \frac{1}{\delta_j}}(s - t), \quad \text{for all } t = 1, \dots, n.$$

Putting $t = n$ in the condition above, we obtain

$$v(s) \geq \left(1 - \frac{n - s}{\sum_{j \in N} \frac{1}{\delta_j}}\right)v(n), \quad (16)$$

since $V_\sigma \leq v(n)$ and $s < n$. Since $\sum_{j \in N} \frac{1}{\delta_j} > n$, (16) implies

$$v(s) > \left(1 - \frac{n - s}{n}\right)v(n) = \frac{s}{n}v(n).$$

This contradicts the assumption that the core of (N, v) is non-empty. Thus, we can prove that the grand-coalition SSPE is a unique symmetric SSPE. QED

Together with Theorem 4.1, the theorem shows that if the proportional distribution of the total value $v(N)$ under the ratio $(p_i/(1 - \delta_i))_{i \in N}$ is in the core, then this is a unique symmetric SSPE allocation.

To conclude this section, we summarize the SSPE payoff in a three-person symmetric game.

Example 4.1. Three-person symmetric games

The player set is $\{1, 2, 3\}$. The value of a coalition S is represented by $v(s)$ where s is the number of players in S . We normalize v as $v(3) = 1$ and

$v(1) = 0$ for $i = 1, 2, 3$. Let $v(2) = a$ where $0 \leq a \leq 1$. The core of the game is non-empty if and only if $0 \leq a \leq 2/3$. Other cooperative solutions such as the Shapley value and the nucleolus are equal to the equity allocation $(1/3, 1/3, 1/3)$, regardless of the value a . For simplicity of analysis, we assume the equal recognition probability ($p_1 = p_2 = p_3 = 1/3$) and the common discount factor ($\delta_1 = \delta_2 = \delta_3 = \delta$).

First, consider the grand-coalition SSPE. By Theorem 4.1, every player receives the expected payoff $1/3$, and thus the grand-coalition SSPE exists if and only if $a \leq \frac{3-\delta}{3}$ (see (8)). Next, consider a Baron-Ferejohn equilibrium in which every player chooses every two-person coalition including herself at random. It follows from Proposition 4.2 that every player receives the expected payoff $a/3$, and that the Baron-Ferejohn equilibrium exists if and only if $\frac{3}{3+\delta} \leq a \leq 1$ (putting $t = 3$ in (15)).

For the range $\frac{3-\delta}{3} < a < \frac{3}{3+\delta}$, we will show that there exists a symmetric SSPE in which every player chooses both the grand coalition and all two-person coalitions including herself at random. Suppose that every player chooses the grand coalition with probability p ($0 < p < 1$) and each of two-person coalitions including herself with probability q ($0 < q < 1/2$) where $p = 1 - 2q$. Lemma 4.1 shows that every player receives the same expected payoff, denoted by v . The payoff equation (5) of the SSPE is given by

$$v = \frac{1}{3}(p(1 - 2\delta v) + 2q(a - \delta v)) + \frac{2}{3}(p + q)\delta v.$$

This solves $p = \frac{3v-a}{1-a}$. On the other hand, the optimality condition (4) of the SSPE requires that $1 - 2\delta v = a - \delta v$, which implies $v = \frac{1-a}{\delta}$. Thus, we have $p = \frac{3-(3+\delta)a}{(1-a)\delta}$. The constraint $0 < p < 1$ means that $\frac{3-\delta}{3} < a < \frac{3}{3+\delta}$.

The expected payoff v of every player is illustrated in Figure 4.1 when the discount factor for future payoffs is almost equal to one. The bargaining outcome depends on the value a of a two-person coalition as follows. When $0 \leq a \leq \frac{3-\delta}{3}$, the grand coalition forms and v is constant at $1/3$. When $\frac{3-\delta}{3} \leq$

$a \leq \frac{3}{3+\delta}$, it is interesting to notice that the expected payoff decreases as the value of two-person coalitions increases. In this region, every player proposes all coalitions including herself at random, and the probability of the grand coalition decreases as two-person coalitions become more valuable. When $a = \frac{3}{3+\delta}$, the grand coalition is never chosen, and thereafter v increases as the value of two-person coalitions increases, and becomes $1/3$ again when $a = 1$ (the majority game).

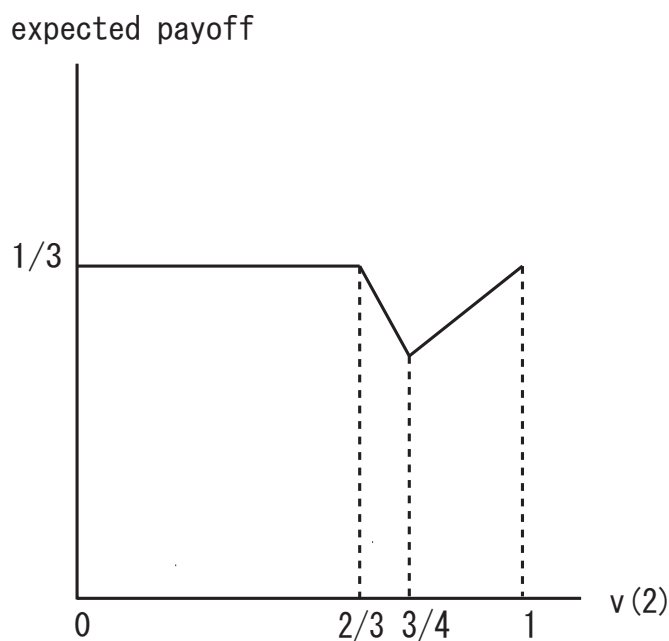


Figure 4.1 three-person symmetric game

5 Application: one employer and workers

Consider a production economy \mathcal{E} with one employer (player 1) and $n - 1$ identical workers i ($= 2, \dots, n$). Let $N = \{1, \dots, n\}$. If a coalition $S \subset N$

consists of the employer and $s-1$ ($s \geq 1$) workers, then the benefit of S is given by a real-valued function $f(s)$ which is monotonically increasing in $s = 1, \dots, n$ with $f(1) = 0$. Otherwise, the benefits of coalitions are assumed to be zero. Shapley and Shubik (1967) investigated this economy by cooperative game theory. The core of the economy is always non-empty since the allocation with the employer exploiting the total benefit $f(n)$ is in the core. The Shapley value and the nucleolus in the economy will be given in Proposition 5.4.

The central questions in the production economy are: how many workers does the employer hire, under what conditions is the efficient outcome of full-employment attained, and how much wages do workers receive? To answer these questions, we will apply the random proposer model $\Gamma(N, p, \delta)$ to the production economy \mathcal{E} , and will characterize an SSPE of the economy. For simplicity of analysis, we will assume that all players have the common discount factor δ and the equal recognition probability $p = (1/n, \dots, 1/n)$. Let v_i be the expected payoff of player i ($i = 1, \dots, n$) in an SSPE. The grand-coalition SSPE will be called the *full-employment equilibrium*, and any other SSPE a *partial-employment equilibrium*.

The first result regarding the full employment is derived by Theorem 4.1.

Proposition 5.1. The full-employment equilibrium of the production economy \mathcal{E} is characterized as follows.

(i) The full-employment equilibrium exists if and only if

$$\frac{n - (n - s)\delta}{n} f(n) \geq f(s) \quad \text{for all } s = 1, \dots, n. \quad (17)$$

(ii) The expected payoff v_i of every player i ($i = 1, \dots, n$) is given by $f(n)/n$.

Every proposer offers $\delta f(n)/n$ to all other players, and it is accepted.

When the discount factor δ is almost equal to one, condition (17) is equivalent to $f(n)/n \geq f(s)/s$ for all $s = 1, \dots, n$. This indicates that the full

employment has the highest average productivity. The proposition shows that all players receive the same expected payoffs in the full employment equilibrium. When they are sufficiently patient, the employer and all workers agree to the equal allocation.

We next characterize a partial-employment equilibrium where there exist some workers unemployed. The next lemma shows that all workers receive the same expected payoffs in every SSPE.

Lemma 5.1. For all workers i and j , $v_i = v_j$ for every SSPE.

The proof of the lemma is given in the Appendix. We define the following two types of partial-employment equilibria.

Definition 5.1. Let σ be a partial-employment equilibrium of the production economy \mathcal{E} .

- (i) For $2 \leq s < n$, σ is called an *s-equilibrium* if only coalitions with s members form with positive probability in σ .
- (ii) For $2 \leq s < t \leq n$, σ is called an *(s,t)-equilibrium* if only coalitions with s and t members form with positive probability in σ .

In an *s-equilibrium*, the employer hires only $s - 1$ workers. In an *(s,t)-equilibrium*, the employer hires randomly either $s - 1$ or $t - 1$ workers. There exist no other types of partial-employment equilibria except a degenerate class of the economy \mathcal{E} (see footnote 5).

Proposition 5.2. For $2 \leq s < n$, an *s-equilibrium* of the production economy \mathcal{E} is characterized as follows.

- (i) Let $v(\delta)$ be the expected payoff of the employer, and let $w(\delta)$ be the

expected payoff of every worker. Then,

$$v(\delta) = \frac{n-1-(s-1)\delta}{(n-1)\{n-(n-1)\delta\}-(s-1)\delta} f(s) \quad (18)$$

$$w(\delta) = \frac{(n-1)(1-\delta)}{(n-1)\{n-(n-1)\delta\}-(s-1)\delta} f(s). \quad (19)$$

Every worker receives an offer with probability $\frac{(s-2)n+1}{n(n-1)}$.

- (ii) There exists an s -equilibrium if and only if $f(s) - f(t) \geq (s-t)\delta w(\delta)$ for all $t \neq s$.
- (iii) An s -equilibrium exists for any δ sufficiently close to one if and only if $f(s) > f(t)$ for all $t < s$ and $f(s) = f(t)$ for all $t > s$. As δ goes to one, the equilibrium allocation converges to a unique core-allocation of the economy \mathcal{E} where the employer exploits the total payoff $f(n)$.

The worker's equilibrium wage must be equal to the continuation payoff $\delta w(\delta)$ in the random-proposer model. Thus, the employer's profit maximization is given by

$$\max_k f(k) - (\delta w(\delta))(k-1)$$

where $k-1$ is the number of hired workers. In an s -equilibrium, the employer's profit is maximized at $k = s$. The existence condition (ii) for an s -equilibrium in our discrete case corresponds to the well-known condition for profit maximization that marginal productivity be equal to the equilibrium wage, that is, $f'(s) = \delta w(\delta)$, in the case that the number s of workers is a continuous variable.

When the discount factor δ is almost close to one, the proposition shows that the equilibrium wages for workers converge to zero and that the employer exploits the total production $f(n)$. The equilibrium allocation $(f(n), 0, \dots, 0)$ is a unique core allocation in the economy.

The intuition for this result is as follows. The expected payoff $v(\delta)$ of the

employer satisfies the payoff equation

$$v(\delta) = \frac{1}{n}(f(s) - (s-1)\delta w(\delta)) + \frac{n-1}{n}\delta v(\delta).$$

In the limit that δ goes to one, the equation $v^* + (s-1)w^* = f(s)$ holds. That is, when the discount factor δ is sufficiently close to one, the residual claim of the employer is equal to her expected payoff v^* in all s -member coalitions which she may propose. On the other hand, since an s -member coalition forms with probability one in equilibrium, it must be $v^* + (n-1)w^* = f(s)$. These two equations give $w^* = 0$. Then, since it is optimal for the employer to hire $s-1$ workers, it must be $f(s) = f(n)$. Therefore, the equilibrium allocation is $(f(n), 0, \dots, 0)$, which is a unique core allocation.

The next proposition characterizes a partial-employment equilibrium where the employer may hire two different numbers of workers with positive probability.

Proposition 5.3. For $2 \leq s < t \leq n$, an (s, t) -equilibrium of the production economy \mathcal{E} is characterized as follows.

- (i) Let $v(\delta)$ be the expected payoff of the employer, $w(\delta)$ the expected payoff of every worker, and $p(\delta)$ the probability that an s -member coalition is formed. Then,

$$v(\delta) = \frac{(t-1)f(s) - (s-1)f(t)}{\{n - (n-1)\delta\}(t-s)} \quad (20)$$

$$w(\delta) = \frac{f(t) - f(s)}{(t-s)\delta} \quad (21)$$

$$p(\delta) = \frac{f(t) - v(\delta)}{f(t) - f(s)} - \frac{n-1}{(t-s)\delta}. \quad (22)$$

- (ii) There exists an (s, t) -equilibrium if and only if $0 < p(\delta) < 1$ and $f(s) - f(k) \geq (s-k)\delta w(\delta)$ for all $k \neq s$ with equality for $k = t$.
- (iii) Every (s, t) -equilibrium and every (s', t') -equilibrium are payoff-equivalent

if $s, s' < t, t'$.

- (iv) Assume $t < n$. If an (s, t) -equilibrium exists for any δ sufficiently close to one, then $f(k) = f(n)$ for all $s \leq k \leq n$. Moreover, $v(\delta)$ and $w(\delta)$ converge to $f(n)$ and 0, respectively, as δ goes to one.
- (v) Assume $t = n$. If an (s, n) -equilibrium exists for any δ sufficiently close to one, then $\frac{f(n)}{n} \leq \frac{f(s)}{s}$. Moreover, $v(\delta)$ and $w(\delta)$ converge to $v^* = \frac{(n-1)f(s) - (s-1)f(n)}{n-s}$ and $w^* = \frac{f(n) - f(s)}{n-s}$, respectively, as δ goes to one. If $f(s) < f(n)$, then $p(\delta)$ converges to zero as δ goes to one. The expected payoff vector (v^*, w^*, \dots, w^*) is in the core of the production economy \mathcal{E} , and the employer receives the minimum payoff v^* in the core.

In the limit that the discount factor is almost equal to one, the proposition characterizes an (s, t) -equilibrium as follows. When unemployment may occur with probability one ($t < n$), the equilibrium outcome is the same as in an s -equilibrium. That is, the equilibrium wage is zero, and the employer exploits the total production $f(n)$. On the other hand, when full employment may occur with positive probability ($t = n$), the equilibrium wage is $w^* = \frac{f(n) - f(s)}{n-s}$, and the equilibrium allocation (v^*, w^*, \dots, w^*) lies in the core of the economy. The efficiency of allocation can be attained when the employer and all workers are sufficiently patient.

The same intuition as for Proposition 5.2 explains the result of Proposition 5.3 when the discount factor δ is almost equal to one. Since it is optimal for the employer to hire $s - 1$ workers and $t - 1$ workers, it must be $f(s) - (s - 1)w^* = f(t) - (t - 1)w^*$, and thus the equilibrium wage is given by $w^* = \frac{f(t) - f(s)}{t-s}$. Since the residual claim of the employer is equal to her expected payoff v^* in all coalitions which she may propose, it must be $v^* + (s - 1)w^* = f(s)$ and $v^* + (t - 1)w^* = f(t)$. On the other hand, the sum of all players' expected payoffs must satisfy $v^* + (n - 1)w^* = p^*f(s) + (1 - p^*)f(t)$ where p^* is the probability for an s -member coalition. When $t < n$, these equations give

$w^* = 0$ and $v^* = f(s) = f(t)$. $v^* = f(n)$ is derived by the fact that it is optimal for the employer to form s -member coalitions.

When $t = n$, the equilibrium conditions give $w^* = \frac{f(n)-f(s)}{n-s}$, and $v^* + (s-1)w^* = f(s)$ and $v^* + (n-1)w^* = f(n)$. In addition, since it is optimal for the employer to hire all workers, it must be $v^* = f(n) - (n-1)w^* \geq f(k) - (k-1)w^*$ for all k ($2 \leq k \leq n$), equivalently, $v^* + (k-1)w^* \geq f(k)$ for all k . That is, the equilibrium allocation (v^*, w^*, \dots, w^*) lies in the core of the economy. The core property also shows that the equilibrium wage w^* is the solution of

$$\min_{1 \leq k \leq n-1} \frac{f(n) - f(k)}{n - k}. \quad (23)$$

Finally, we remark that the core constraint $v^* + (k-1)w^* \geq f(k)$ is binding at $k = s$. This implies that the equilibrium allocation (v^*, w^*, \dots, w^*) is located on the boundary of the core, and that the employer receives the smallest payoff in the core. Another property of the equilibrium allocation (v^*, w^*, \dots, w^*) is that it maximizes the Nash product $\prod_{i \in N} x_i$ over the core. Compte and Jehiel (2008) call such a solution the *coalitional Nash bargaining solution* and characterize it as the outcome of an asymptotic efficient equilibrium in the random proposer model.

We summarize the equilibrium of the production economy \mathcal{E} when the employer and all workers are sufficiently patient.

Theorem 5.1. The production economy \mathcal{E} has a unique SSPE in terms of expected payoffs when the employer and all workers are sufficiently patient. In the limit that the discount factor δ goes to one, the expected payoffs for players are characterized as follows.

- (i) If $f(n)/n \geq f(s)/s$ for all s , then all players receive the equal payoff $f(n)/n$. The payoff distribution is in the core of the economy.
- (ii) Otherwise, the workers receive the wage $w^* = \frac{f(n)-f(s)}{n-s}$ where s is the solution of $\min_{1 \leq k \leq n-1} \frac{f(n)-f(k)}{n-k}$. The employer maximizes the profit by

hiring $s-1$ workers. The payoff distribution is in the core of the economy, and the employer receives the smallest payoff in the core.

The next example of a three-person economy illustrates what types of SSPE exist over the whole range of the discount factor δ .

Example 5.1. One employer and two workers

Consider an economy with one employer and two workers. The production function is given by $f(1) = 0$, $f(2) = a$ ($0 \leq a \leq 1$) and $f(3) = 1$. Figure 5.1 illustrates the regions in which each type of equilibria exists, depending on the productivity a of a two-member coalition and the discount factor δ . Proposition 5.1(i) shows that the full-employment equilibrium exists if and only if $\frac{3-\delta}{3} \geq a$ (region A). In region A, all players receive the same expected payoffs $1/3$. Proposition 5.2(ii) shows that a 2-equilibrium exists if and only if $a \geq \frac{6-5\delta}{6-3\delta-2\delta^2}$ (region C). In region C, the employer receives the expected payoff $\frac{2-\delta}{6-5\delta}a$, and the workers receive $\frac{2-2\delta}{6-5\delta}a$. A (2, 3)-equilibrium exists in region B, where the employer receives the expected payoff $\frac{2a-1}{3-2\delta}$, and the workers receive the expected payoff $\frac{1-a}{\delta}$. In region B, the probability of full employment converges to one as the discount factor goes to one. In the limit, the payoff to the employer is $2a - 1$ if $2/3 < a \leq 1$.

We have shown that the SSPE payoff always belongs to the core of the economy in the limit that the discount factor δ is almost equal to one. To conclude our analysis, we compare the SSPE payoff with the Shapley value and the nucleolus. Since the definitions of these cooperative solutions are standard, we omit them.

Proposition 5.4. The Shapley value and the nucleolus in the production economy \mathcal{E} are characterized as follows.

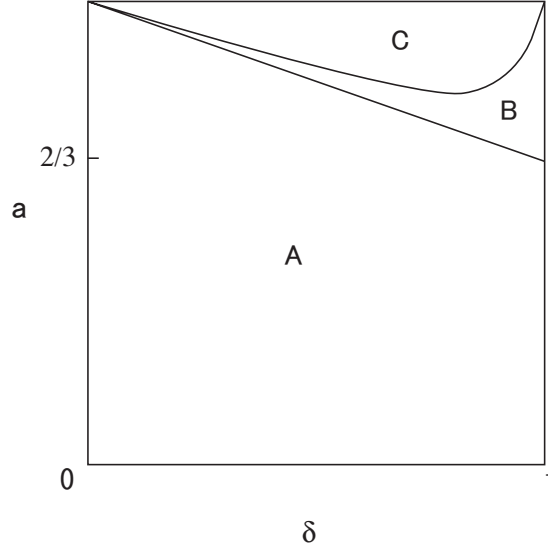


Figure 5.1 One employer and two workers

(i) The Shapley value of the employer is

$$\phi = \sum_{s=1}^n \frac{1}{n} f(s). \quad (24)$$

(ii) The wage w for workers in the nucleolus is

$$\min_{1 \leq s \leq n-1} \frac{f(n) - f(s)}{n - s + 1}. \quad (25)$$

The payoff of the employer is

$$\mu = \frac{(n-1)f(s^*) - (s^* - 2)f(n)}{n - s^* + 1}$$

where s^* is the solution of (25).

The intuition for the Shapley value is as follows. When the employer may enter a randomly forming coalition, she joins any s -member coalition with equal probability because $\binom{n-1}{s-1} \frac{(s-1)!(n-s)!}{n!} = \frac{1}{n}$. The marginal contribution of the employer to the coalition is $f(s)$. Since the Shapley value is defined to be the average of marginal contributions, it is given by (24). The Shapley value and the SSPE payoff are different in several respects. First, the Shapley value does not necessarily belong to the core. Second, the Shapley value is not equal to the equal allocation except the special case that production is possible only under the full-employment. Third, the Shapley value takes into account the values of all possible coalitions. These properties of the Shapley value is due to the random formation of coalitions underlying it. On the contrary, the SSPE captures a strategic conflict in coalition formation.

Unlike the Shapley value, the nucleolus and the SSPE payoff have similar properties. The equal division is realized in the nucleolus if and only if

$$\frac{f(n)}{n} = \min_{1 \leq s \leq n-1} \frac{f(n) - f(s)}{n - s + 1}.$$

This condition implies that $\frac{f(n)}{n} \geq \frac{f(s)}{s-1}$ for $s = 2, \dots, n-1$, which is slightly stronger than the condition $\frac{f(n)}{n} \geq \frac{f(s)}{s}$ for the full-employment equilibrium. When the nucleolus and the SSPE have different wages, Proposition 5.3 shows that the nucleolus wage is higher than the SSPE wage. However, as we can see from (23) and (25), the difference between the wages in the two solutions is marginal. From these observations, we conclude that the nucleolus and the SSPE have similar distributional properties in the production economy.

The example of a three-person economy is helpful for us to compare further the three solutions.

Example 5.2. Comparison of non-cooperative and cooperative solutions

Consider again the economy with one employer and two workers in Example 5.1. The production function is given by $f(1) = 0, f(2) = a$ ($0 \leq a \leq 1$) and

$f(3) = 1$. We assume that the discount factor δ is almost equal to one. Let v be the employer's payoff and w the wage for workers. All solutions are efficient, that is, $v + 2w = 1$. The Shapley value satisfies $v = \frac{1+a}{3}$. The nucleolus has the wage $w = \min(\frac{1}{3}, \frac{1-a}{2})$. The SSPE has the wage $w = \min(\frac{1}{3}, 1-a)$. Figure 5.2 illustrates the employer's payoff in the three solutions over the whole range of parameter a . When $0 \leq a \leq 1/3$, the SSPE payoff and the nucleolus, both included in the core, give the equal division. When $1/3 \leq a \leq 2/3$, the nucleolus wage is $(1-a)/2 (< 1/3)$ but the SSPE wage is still $1/3$. When $2/3 \leq a \leq 1$, the SSPE wage becomes $1-a$. The SSPE payoff allocation is equal to the equity allocation $(1/3, 1/3, 1/3)$ as long as the latter belongs to the core. The nucleolus, however, departs earlier from the equity allocation as the coalition of the employer and one worker becomes productive ($1/3 < a$). As we can see in Figure 5.2, the Shapley value behaves differently from the nucleolus and the SSPE. The wage of the Shapley value decreases as the two-member coalition becomes productive, but the magnitude of the decrease is not as steep as the nucleolus and the SSPE. It gives each worker the payoff $1/6$ even when the employer can make the highest production with only one worker. However, in this case, the core, the nucleolus and the SSPE all predict the employer's exploitation. Figure 5.3 illustrates the locations of different solutions over the range of $2/3 < a < 4/5$ in the set of payoff distributions (x_1, x_2, x_3) which satisfy $x_1 + x_2 + x_3 = 1$. The payoff $x_i (i = 1, 2, 3)$ for player i is given by the distance between the allocation x and the edge jk .

6 Conclusion

A non-cooperative bargaining model with random proposers can serve as a bridge connecting two branches of game theory, non-cooperative games and cooperative games. The model is a natural generalization of Rubinstein's two-person alternating-offers model. It is not only tractable but also rich enough to relate a non-cooperative bargaining equilibrium to the classical cooperative

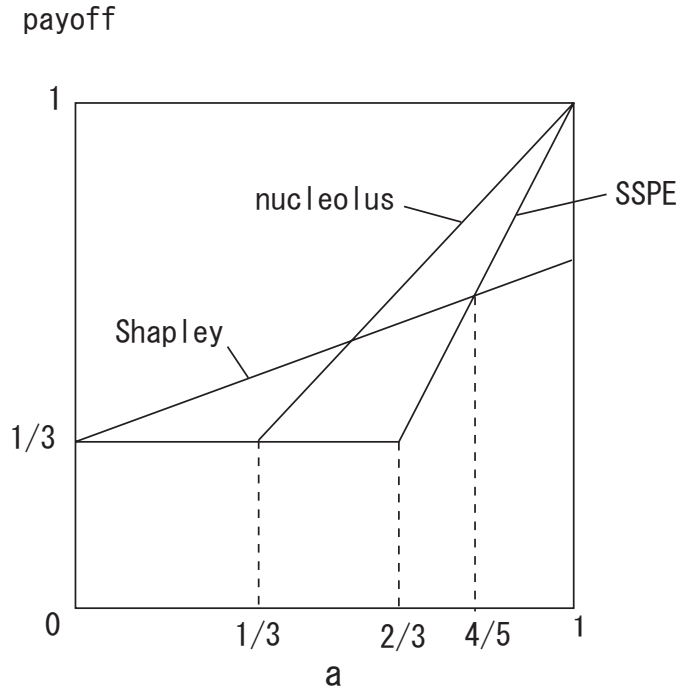


Figure 5.2 The payoff of the employer in three solutions

solutions such as the core, the Nash bargaining solution and the nucleolus. The analysis of a production economy has shown how unemployment may occur related to agents' productivity and their time preference. It would be an interesting work to analyze the model further in other economic situations for future research.

7 Appendix

Proof of Lemma 5.1. The lemma is proved by Yan (2005, Proposition 5). For convenience of readers, we provide her proof here. Let \mathcal{C}_i be the set of coalitions which player i proposes with positive probability in an SSPE, and let q_i^j be the probability that player i receives an offer when player j is selected

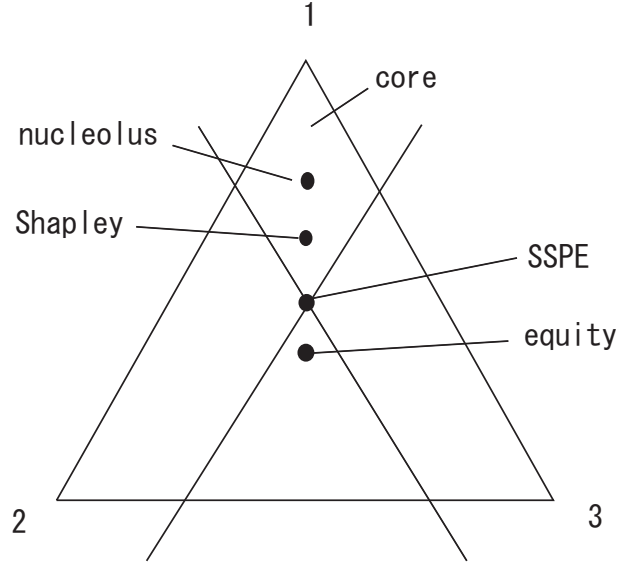


Figure 5.3 Comparison of different solutions

as a proposer. For any S including i , define $w(S) = v(S) - \sum_{j \in S} \delta v_j$. Then, it must hold from Lemma 3.2 that

$$v_i = \frac{1}{n}(w(S) + \delta v_i) + \frac{1}{n} \sum_{j \neq i} q_i^j \delta v_j \quad (26)$$

for all $S \in \mathcal{C}_i$, and

$$w(S) \geq w(T) \quad (27)$$

for all $S \in \mathcal{C}_i$ and all $T \subset N$ with $i \in T$. (26) implies

$$v_i = \frac{w(S)}{n - \delta(1 + \sum_{j \neq i} q_i^j)} \quad \text{for all } S \in \mathcal{C}_i. \quad (28)$$

Remark from (27) that $w(S) = w(S')$ for all $S, S' \in \mathcal{C}_i$.

Suppose $v_i \neq v_j$. With no loss of generality, we assume $v_i > v_j$.

Claim 1. $w(S^j) \geq w(S^i)$ for all $S^j \in \mathcal{C}_j$ and all $S^i \in \mathcal{C}_i$.

Proof of claim 1. Take any $S^j \in \mathcal{C}_j$ and any $S^i \in \mathcal{C}_i$. If $j \in S^i$, then the claim follows from (27). Otherwise, we have $w(S^j) \geq w((S^i - \{i\}) \cup \{j\}) = w(S^i) + \delta v_i - \delta v_j > w(S^i)$.

Claim 2. $\sum_{k \neq j} q_j^k \geq \sum_{k \neq i} q_i^k$.

Proof of claim 2. Consider any $k \neq i, j$. For any $S \in \mathcal{C}_k$, $i \in S$ implies $j \in S$ since $\delta v_i > \delta v_j$. Hence, $q_j^k \geq q_i^k$. We next prove $q_j^i \geq q_i^j$. It suffices us to show that $q_j^i < 1$ implies $q_i^j = 0$, that is, if there exists some $S^i \in \mathcal{C}_i$ with $j \notin S^i$, then any $S^j \in \mathcal{C}_j$ does not include i . Suppose not. Then, there exists some $S^i \in \mathcal{C}_i$ with $j \notin S^i$ and some $S^j \in \mathcal{C}_j$ with $i \in S^j$. Since $S^i \in \mathcal{C}_i$ and $i \in S^j$, we have $w(S^i) \geq w(S^j) > w(S^j) + \delta v_j - \delta v_i$. On the other hand, since $S^j \in \mathcal{C}_j$, we have $w(S^j) \geq w((S^i - \{i\}) \cup \{j\}) = w(S^i) + \delta v_i - \delta v_j$. A contradiction.

By (28), the two claims imply $v_j \geq v_i$. This contradicts the supposition. QED

Proof of Proposition 5.2.

(i) It follows from Lemma 3.2 that

$$\begin{aligned} v(\delta) &= \frac{1}{n}(f(s) - (s-1)\delta w(\delta)) + \frac{n-1}{n}\delta v(\delta) \\ f(s) &= v(\delta) + (n-1)w(\delta). \end{aligned}$$

It can be easily shown that this system has a unique solution (18) and (19). Let q be the probability that every worker receives an offer. Again, from Lemma 3.2, we have

$$w(\delta) = \frac{1}{n}(f(s) - \delta v(\delta) - (s-2)\delta w(\delta)) + q\delta w(\delta).$$

Together with (18) and (19), we can prove that this equation has a solution $q = \frac{(s-2)n+1}{n(n-1)}$.

(ii) The optimality condition (4) for the employer yields

$$f(s) - (s - 1)\delta w(\delta) \geq f(t) - (t - 1)\delta w(\delta) \quad \text{for all } t \neq s.$$

This is equivalent to $f(s) - f(t) \geq (s - t)\delta w(\delta)$ for all $t \neq s$. The optimality conditions for workers give the same inequality. Conversely, we can construct an s -equilibrium as follows. The employer selects every $s - 1$ workers with equal probability, and offers $\delta w(\delta)$ to them. Every worker selects every s -member coalition including the employer and herself with equal probability, and offers $\delta v(\delta)$ to the employer, and $\delta w(\delta)$ to other workers in the coalition. When this strategy is employed, every worker receives an offer with probability $\frac{(s-2)n+1}{n(n-1)}$. If the condition in the proposition holds, then we can prove from Lemma 3.2 that the constructed strategy is an SSPE.

(iii) In view of (19), we can see from $s < n$ that $w(\delta)$ converges to zero as δ goes to one. Noting this fact, we can show that for all t ,

$$f(s) - f(t) \geq (s - t)\delta w(\delta) \quad \text{for any } \delta \text{ sufficiently close to one}$$

if and only if

$$f(s) \geq f(t) \quad \text{for all } t = 1, \dots, n$$

with the strict inequality for all $t < s$. Since $f(s)$ is a monotonically increasing function, this proves the first part. By (18) and (19), we have

$$\lim_{\delta \rightarrow 1} v(\delta) = f(n) \quad \text{and} \quad \lim_{\delta \rightarrow 1} w(\delta) = 0.$$

Hence, the employer offers zero payoff to every worker in a coalition in the limit that δ goes to one. When $f(s) = f(t)$ for all $t \geq s$, the core of the production economy \mathcal{E} consists of a unique allocation $(f(n), 0, \dots, 0)$. This proves the second part. QED

Proof of Proposition 5.3.

(i) By Lemma 3.2, $v = v(\delta)$ and $w = w(\delta)$ satisfy the following conditions.

$$v = \frac{1}{n}(f(s) - (s-1)\delta w) + \frac{n-1}{n}\delta v \quad (29)$$

$$v + (n-1)w = pf(s) + (1-p)f(t) \quad (30)$$

$$f(s) - (s-1)\delta w \geq f(k) - (k-1)\delta w \text{ for all } k \neq s \quad (31)$$

$$f(t) - (t-1)\delta w \geq f(k) - (k-1)\delta w \text{ for all } k \neq t \quad (32)$$

By (31) and (32), we have $f(s) - (s-1)\delta w = f(t) - (t-1)\delta w$, which yields (21). By substituting (21) into (29), we have (20). Finally, we obtain (22) from (21) and (30).

(ii) (31) and (32) are equivalently reduced to $f(s) - f(k) \geq (s-k)\delta w(\delta)$ for all $k \neq s, t$ and $f(s) - f(t) = (s-t)\delta w(\delta)$. The proposition can be proved by Lemma 3.2.

(iii) Let v and w be the expected payoffs for the employer and workers, respectively, in an (s, t) -equilibrium, and let v' and w' be the expected payoffs for the employer and workers, respectively, in an (s', t') -equilibrium. In view of (21), the optimality conditions for an (s, t) -equilibrium are

$$f(s) - f(k) \geq (s-k) \frac{f(t) - f(s)}{t-s} \quad (33)$$

$$f(t) - f(k) \geq (t-k) \frac{f(t) - f(s)}{t-s}. \quad (34)$$

for any $k \neq s, t$. Similarly, the optimality conditions for an (s', t') -equilibrium are

$$f(s') - f(k) \geq (s'-k) \frac{f(t') - f(s')}{t'-s'} \quad (35)$$

$$f(t') - f(k) \geq (t'-k) \frac{f(t') - f(s')}{t'-s'} \quad (36)$$

for any $k \neq s', t'$. Putting $k = s'$ in (34) and $k = t$ in (35), we obtain

$$\frac{f(t) - f(s)}{t-s} \leq \frac{f(t') - f(s')}{t'-s'}$$

since $s' < t$. Similarly, putting $k = t'$ in (33) and $k = s$ in (36), we obtain

$$\frac{f(t) - f(s)}{t - s} \geq \frac{f(t') - f(s')}{t' - s'}$$

since $s < t'$. Thus, we have $w = w'$. If $s = s'$, then $v = v'$ easily follows from (29) and $w = w'$. Suppose that $s > s'$, without loss of generality. Putting $k = s'$ in (33), we have

$$\frac{f(s) - f(s')}{s - s'} \geq \delta w.$$

Similarly, putting $k = s$ in (35), we have

$$\frac{f(s) - f(s')}{s - s'} \leq \delta w'.$$

Hence, we have

$$\frac{f(s) - f(s')}{s - s'} = \delta w = \delta w'.$$

This equality with (29) yields $v = v'$.

(iv) From (20) and (21), we can see that $v(\delta)$ and $w(\delta)$ converge to

$$v^* = \frac{(t-1)f(s) - (s-1)f(t)}{t-s}, \quad w^* = \frac{f(t) - f(s)}{t-s}, \quad (37)$$

respectively, as δ goes to one. Let p^* be any accumulation point of $\{p(\delta)\}$. By taking $\delta \rightarrow 1$ in (30), we have $v^* + (n-1)w^* = p^*f(s) + (1-p^*)f(t)$. Substituting (37) into this equation, we obtain

$$\left(p^* + \frac{n-t}{t-s}\right)f(s) = \left(p^* + \frac{n-t}{t-s}\right)f(t). \quad (38)$$

Since $t < n$, it must be $p^* + \frac{n-t}{t-s} > 0$. Then, it follows from (38) that $f(s) = f(t)$. Thus, $v^* = f(s)$ and $w^* = 0$ from (37). Finally, $f(s) = f(t)$ implies $f(s) = f(n)$ from (33) with $k = n$.

(v) When $t = n$, we have $v^* + (n-1)w^* = p^*f(s) + (1-p^*)f(n)$. Putting $t = n$ in (37) yields $v^* = \frac{(n-1)f(s) - (s-1)f(n)}{n-s}$ and $w^* = \frac{f(n) - f(s)}{n-s}$. Since $t = n$,

(38) implies that $p^*f(s) = p^*f(n)$ for any accumulation point p^* of $\{p(\delta)\}$. If $f(s) < f(n)$, then we obtain $p^* = 0$. Hence, $\{p(\delta)\}$ converges to 0 as δ goes to one. Then, whichever $f(s) < f(n)$ or $f(s) = f(n)$ holds, we obtain

$$v^* + (n - 1)w^* = f(n). \quad (39)$$

Since $p \geq 0$ and $f(s) \leq f(n)$, (30) implies $v(\delta) + (n - 1)w(\delta) \leq f(n)$. Substituting (20) and (21) into this inequality, a tedious calculation yields

$$\frac{f(s)}{n - (n - s)\delta} \geq \frac{f(n)}{n}.$$

By taking $\delta \rightarrow 1$, we obtain $\frac{f(s)}{s} \geq \frac{f(n)}{n}$. We now prove the last part. It follows from (37) that

$$v^* + (k - 1)w^* = \frac{(n - k)f(s) - (s - k)f(n)}{n - s} \quad (40)$$

for all $k \geq 2$. On the other hand, (34) yields

$$f(n) - f(k) \geq (n - k)\frac{f(n) - f(s)}{n - s}. \quad (41)$$

(40) and (41) imply that $v^* + (k - 1)w^* \geq f(k)$ for all $k \geq 2$. Together with (39), this means that the payoff allocation (v^*, w^*, \dots, w^*) is in the core of the production economy \mathcal{E} . By setting $k = s$ in (40), we have $v^* + (s - 1)w^* = f(s)$. Any core allocation (v_1, w, \dots, w) in which all workers receive the same payoffs w satisfies $v + (n - 1)w = f(n)$ and $v + (s - 1)w \geq f(s)$. These conditions imply $v \geq v^*$. Finally, for any core allocation (v_1, w_2, \dots, w_n) , there exists some value w such that the allocation (v_1, w, \dots, w) is in the core. Thus, the employer's payoff v^* is her minimum payoff in the core. QED

Proof of Theorem 5.1. We first remark that even if there exists an SSPE with more than two different sizes of coalitions, the same proof as in Propo-

sition 5.3 can be applied. Specifically, the equilibrium wage $w(\delta)$ satisfies $f(s) - f(t) = (s - t)\delta w(\delta)$ for all coalitional sizes s and t which may occur in equilibrium.⁶ This implies that the economy has the same payoff distribution as in an (s, t) -equilibrium in the limit that the discount factor δ goes to one. Noting this fact, it follows from Propositions 5.1, 5.2 and 5.3 that the economy has a unique SSPE payoff when the discount factor δ is close to one. If $f(n)/n \geq f(s)/s$ for all s ($1 \leq s \leq n$), Proposition 5.1 shows that there exists the full-employment equilibrium, and that the equilibrium allocation $(f(n)/n, \dots, f(n)/n)$ is in the core. Thus, (i) holds. Suppose that there exists some k ($1 \leq k \leq n - 1$) such that $f(n)/n < f(k)/k$, which implies $\frac{f(n)-f(k)}{n-k} < \frac{f(n)}{n}$. Let s be the solution of $\min_{1 \leq k \leq n-1} \frac{f(n)-f(k)}{n-k}$ and let $w = \frac{f(n)-f(s)}{n-s}$. By the definitions of s and w , we can show that $f(n)-(n-1)w \geq f(k)-(k-1)w$ for any k . The employer's profit $f(k)-(k-1)w$ is maximized at $k = s, n$. Then, (ii) is proved by Propositions 5.2 and 5.3. QED.

Proof of Proposition 5.4.

(i) By the definition of the Shapley value, we have

$$\phi_1 = \sum_{s=1}^n \binom{n-1}{s-1} \frac{(s-1)!(n-s)!}{n!} f(s) = \sum_{s=1}^n \frac{1}{n} f(s). \quad (42)$$

(ii) Since the nucleolus satisfies the axiom of symmetry, all (identical) workers receive the same payoffs. Let v be the employer's payoff and w the wage for

⁶Since the ratio $\frac{f(s)-f(t)}{s-t}$ is constant for any two coalition sizes s and t in equilibrium, it is generically true that the economy does not possess any SSPE with more than two coalition sizes, given the discount factor δ .

workers. Then, the nucleolus is the solution of the following program:

$$\begin{aligned}
& \min \quad \varepsilon \\
& \text{s.t.} \quad v + (s - 1)w + \varepsilon \geq f(s), \quad s = 1, \dots, n - 1 \\
& \quad \quad \quad sw + \varepsilon \geq 0, \quad s = 1, \dots, n - 1 \\
& \quad \quad \quad v + (n - 1)w = f(n)
\end{aligned}$$

This program has a feasible solution: $v = f(n), w = 0, \varepsilon = 0$. Let $(v^*, w^*, \varepsilon^*)$ denote the optimal solution. If $w^* < 0$, then $\varepsilon^* \geq -w^* > 0$. A contradiction. Thus, it must be $w^* \geq 0$. Now, $n - 2$ constraints $sw + \varepsilon \geq 0$ for $s = 2, \dots, n - 1$ become redundant. The remaining constraints imply

$$\varepsilon \geq -\frac{f(n) - f(s)}{n - s + 1}, \quad s = 1, \dots, n - 1.$$

We can show that the optimal solution $(v^*, w^*, \varepsilon^*)$ satisfies

$$w^* = -\varepsilon^* = \min_{1 \leq s \leq n-1} \frac{f(n) - f(s)}{n - s + 1}, \quad v^* = f(n) - (n - 1)w^*.$$

This proves the proposition. QED

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