Incomplete Financial Markets, Irreversibility of Investments, and Fiscal and Monetary Policy Instruments

KENJI MIYAZAKI ‡ KIYOHIKO G. NISHIMURA § MAKOTO SAITO ¶

February 2008

ABSTRACT. In this paper, we analyze the use of fiscal and monetary instruments to improve long-run welfare when productive investment is irreversible and uncollateralizable and there is no insurance. Only fiat money or government-issued bonds provide self-insurance. We demonstrate that an increase in precautionary savings reduces irreversible productive investment. Hence, subsidies to promote productive but irreversible investment should be financed in such a way that they do not reduce insurance capability. When lump-sum subsidies are high, a consumption tax is likely to be more redistributive and thus more consumption smoothing than are the other sets of instruments analyzed in our model.

JEL classification: D52, D81, H21.
Keywords: fiscal instruments, irreversibility, incomplete markets, liquidity constraints, risk sharing.

* Correspondence to: Kenji Miyazaki, Faculty of Economics, Hosei University, 4342 Aihara, Machida, Tokyo, Japan, 194-0298; e-mail: miya_ken@hosei.ac.jp; tel: +81-42-783-2591; fax: +81-42-783-2611
† We would like to thank an anonymous referee, Koichi Hamada, Yasushi Iwamoto, Akihisa Shibata, and seminar participants at the Economic and Social Research Institute, Hitotsubashi University and Hosei University for their thoughtful and helpful comments. We would also like to acknowledge financial support from the Cabinet Office of the Government of Japan, and a Grant-in-Aid from the Ministry of Education of the Government of Japan. The views expressed here are the personal views of the authors and in no way represent the views of the institutions to which the authors belong.
‡ Hosei University
§ Bank of Japan
¶ Hitotsubashi University
1. Introduction  In this paper, we analyze the use of fiscal and monetary instruments for improving long-run welfare when financial markets are incomplete and economic agents’ incomes are not perfectly observable. Markets are incomplete in the sense that productive investment is irreversible and collateralizable, and there is no insurance against unobservable idiosyncratic shocks. Instead of insurance contracts, only government-issued bonds or fiat money can serve as precautionary savings devices. Such incompleteness of financial markets and imperfection of income observability are often found in market economies. In particular, we focus on fiscal and monetary instruments that can be used in this type of economy to improve social welfare by promoting productive but irreversible investment without adversely affecting the risk-sharing capabilities of consumers.

In this economy, economic agents may not undertake irreversible investments, even if they are productive. Rather, they may hold more liquid, but less profitable, government-issued securities for precautionary savings. Government-issued securities are not backed directly by productive assets but are circulated between generations. Thus, a shift of financial funds from productive investment to government-issued securities may reduce consumption opportunities in the long term. We examine how subsidies should be granted to those who make irreversible investment decisions, and how these subsidies should be financed. This will shed light on how profitable, but irreversible, investment should be promoted to improve long-run welfare.

Subsidies can either be in the form of lump-sum transfers or can be proportional to the amount of investment. Moreover, subsidies can be financed by lump-sum taxes, consumption taxes, or seigniorage revenues. We do not consider income taxes because of the unobservability of income. Hence, we explore how various combinations of fiscal and monetary policies might be used to remedy this undesirable funds allocation, which favors liquid/unbacked assets at the expense of illiquid/productive assets. Thus, we investigate, analytically and numerically, the welfare effects of these policy instruments, and in particular, the relative desirability of various types of taxation instruments.

To explore the effects of these policy instruments, we use a framework for portfolio choice between liquid and illiquid assets in the context of incomplete markets. We adopt the theoretical framework proposed by Dutta and Kapur (1998) for the following four rea-

---

1 Capital investment characterized by such features may include human capital and intangible assets, such as intellectual property rights.
sons. First, its overlapping generations setup incorporates government-issued securities or fiat money as an instrument of intergenerational transfer. Second, productive investment is assumed to be irreversible and, in addition, uncollateralizable. This assumption implies that investors in productive capital are subject to liquidity constraints. Third, the model is set up in such a way that there is no insurance market. Thus, consumers cannot insure against idiosyncratic shocks directly within their own cohorts. Fourth, under these assumptions, consumers can only partially self-insure against idiosyncratic risks by carrying either government securities or fiat money. Thus, although simple, this framework incorporates the necessary elements of portfolio choice between liquid and illiquid assets in incomplete markets.

An important policy implication of our investigation is that, in examining fiscal and monetary instruments in an economy with irreversible and uncollateralizable productive investment, we should consider not only the promotion of irreversible investment but also the capability of consumers to self-insure. We find that lump-sum subsidies to investors who make irreversible investment commitments should be financed in such a way as to enhance or at least not block the self-insurance capabilities of consumers. In particular, broadly based consumption taxes are more desirable than other instruments because they are less destructive of self-insuring. Large-scale inflationary taxes are not desirable because high rates of inflation make self-insurance through holding money costly and thereby reduce the self-insurance capabilities of consumers. Taxes that are less redistributive than consumption taxes, such as lump-sum taxes, impose disproportionate burdens on consumers experiencing negative income shocks, because these consumers are likely to be subject to liquidity constraints. In this way, lump-sum taxes reduce the self-insurance capabilities of consumers overall. Broadly based consumption taxes may be preferable because these taxes are less burdensome to consumers experiencing negative income shocks (because they are lower spenders) than to consumers experiencing positive income shocks (who are higher spenders). This policy implication is also applicable when there are unobservable preference shocks. In addition, we find that lump-sum subsidies are preferable to proportional ones because they distort portfolio choice between liquid and illiquid assets to a lesser extent.

Our model has differences from and similarities to existing dynamic macroeconomic models with incomplete markets in four aspects.

First, it differs substantially from models with reversible physical capital. According to
Aiyagari (1995), for example, an increase in precautionary savings leads to the accumulation of physical capital. This is because liquid bonds, which are held as a precautionary measure, are backed by physical capital. The reversibility of physical capital makes this possible. In contrast, in our framework of irreversibility, liquid assets are unbacked and circulated between generations; therefore, increasing precautionary savings by holding liquid assets reduces irreversible capital.\(^2\) Given this difference between the two models, the presence of a strong precautionary savings motive may accompany overaccumulation in the physical capital model and underinvestment in our model. For this reason, taxes on capital income may enhance steady-state welfare by discouraging capital accumulation in the former model, whereas subsidizing irreversible investment may improve long-run welfare in our model.

Second, our model is similar to models with incomplete markets in that redistributive taxes help to maintain the risk-sharing capabilities of consumers. Varian (1980) and Eaton and Rosen (1980) examine the desirability of redistributive taxation as social insurance in the presence of uninsured idiosyncratic shocks in a static framework. Subsequent researchers, including Kimball and Mankiw (1989) and Castañeda et al. (2003), use dynamic models to explore the positive implications of redistributive taxes for insurance effects. More recent studies attempt to analyze not only the beneficial but also the detrimental effects of redistributive taxes in a dynamic context. For example, Caucutt et al. (2006) and Conesa and Krueger (2005) consider the distortion arising from labor-supply and investment decisions, and Krueger and Perri (2005) analyze the emergence of tighter constraints on private insurance contracts. Our model of irreversible and uncollateralizable investment is different from these models but shares the same motivation.

Third, our model indicates that financing through seigniorage is not necessarily preferable to financing through taxation. Dutta and Kapur (1998) demonstrate that the benefit of an increase in productive investment exceeds the cost of inflation in the case of small-scale seigniorage in this framework. However, we show that large-scale seigniorage raises the cost of holding money and thereby substantially reduces the risk-sharing capabilities of consumers, particularly high-income earners.\(^3\) In a different context, Bewley (1983) explores the limitations of monetary policy within dynamic models in which fiat money is

\(^2\) In this regard, our model is realistic because government-issued securities serve as major liquid instruments enabling self-insurance in most market economies.

\(^3\) This is particularly important for developing economies because they are often heavily dependent on seigniorage revenues, which may have detrimental welfare consequences.
circulated as a precautionary measure against uninsured idiosyncratic shocks.

Fourth, our model endogenizes the demand for fiat money (government bonds) as a self-insurance device in incomplete markets. Existing models, including those of Ono (1994), Itaya (1998), Krugman (2000), and Matsuzaki (2003), evaluate the effect of fiscal and/or monetary policies when money demand is generated exogenously by either cash-in-advance constraints or money-in-the-utility. In particular, Ono (1994) analyzes a case similar to ours; that is, in Ono’s (1994) model, there emerges a trade-off between money demand and aggregate demand including fixed investment, whereas in our model, there is a trade-off between holding fiat money (government bonds) and long-run irreversible investment.

This paper is organized as follows. In Section 2, we present our framework, which is a modified version of the monetary model proposed by Dutta and Kapur (1998), and we explore policy measures for improving social welfare. In Section 3, we present several numerical examples to assess the qualitative significance of the results obtained in Section 2, especially their global implications. In Section 4, we offer concluding remarks. Proofs of propositions and other technical details are provided in the appendices.

2. Theoretical Framework

In this section, we present a simple monetary model with incomplete markets based on Dutta and Kapur (1998), and derive the positive and normative implications of various policy combinations. In most cases, we establish theoretical propositions on global properties. We also establish some theoretical propositions on local cases in which government-financed subsidies to those who make irreversible investments are minimal. The robustness of these local propositions is examined by using numerical examples in the next section. We then compare the welfare implications of various combinations of fiscal and monetary instruments.

2.1. Basic framework

Consider an economy of overlapping generations of investor–consumers that consists of three cohorts, young, middle-aged, and old. The population mass of each cohort is constant over time and is normalized to unity. An infinite sequence of generations allows for the issuance of government securities, including fiat money, which are handed over from one generation to another.

The young generation is endowed with $y_0$ units of goods, whereas the middle-aged generation suffers an independently and identically distributed (i.i.d.) income shock, which is explained in the next paragraph. There is no endowment or income shock for the old generation.
Let us explain the middle-aged income shock in detail. In particular, we assume that middle-aged income is $y_h$ with probability $\frac{1}{2}$ and is $y_l$ with probability $\frac{1}{2}$, where $y_h > y_l$. Moreover, these i.i.d. income shocks are assumed to be unobservable and to form part of the private information of each consumer of this generation. Because of the unobservability of middle-aged income risks, no standard claim that is contingent on these idiosyncratic shocks is traded in financial markets.

We depart from the original framework of Dutta and Kapur by assuming that idiosyncratic risks are unobservable income shocks rather than unobservable preference shocks.\(^4\) This change makes our exposition simple and transparent without substantially affecting the model. This assumption does not necessarily imply that individual income is not observable at all. A part of individual income may be observable even to outsiders, and this observable part may be shared completely among ex ante identical consumers through the use of standard insurance contracts. Hence, this setup should be interpreted as one that abstracts from an unobservable part of the individual income process.

The difference between unobservable income shocks and preference shocks is not as substantial as it appears. This is because, in our model, income is interpreted as consumable income after deducting expenses to offset unexpected and uninsured negative preference shocks, such as health problems. In this case, income may be observable, but the true expenses that offset negative shocks are not reliably observable. This implies that consumable income, which is net of these expenses, is unlikely to be observable or taxable.\(^5\)

Let us now consider the productive opportunities open to investor–consumers. Following Dutta and Kapur, we assume that only the young generation can undertake productive investment. One unit of investment yields $1 + r$ with certainty two periods later, where $r$ is assumed to be positive ($r > 0$). However, this investment is assumed to be neither reversible nor collateralizable when these investors become middle aged in the next period.\(^6\) Under

\(^4\) We follow Saito and Takeda (2006).

\(^5\) The policy implications of our model do not depend on whether the unobservable components are preference shocks or income shocks. As we demonstrate later, the advantage of consumption taxes arises from unobservable components being reflected in the level of consumption. Preference shocks, such as health condition, may not be observable, but they have direct effects on consumption levels. That is, negative (positive) preference shocks result in less (more) consumption. As in the case of unobservable income shocks, imposing taxes that are proportional to such consumption would have redistributive effects by transferring resources from consumers with positive preference shocks to those with negative shocks.

\(^6\) As explained by Dutta and Kapur (1998), this assumption is reasonable if actions undertaken by young investors are unobservable.
this assumption, the middle aged cannot sell physical capital to the young, who have to wait one more period to gain from it. This implies that the illiquid asset is nontransferable and provides no self-insurance against middle-aged idiosyncratic shocks. Henceforth, let $I$ denote the level of investment undertaken by a representative young investor.

In this framework, insurance contracts (standard contingent claims) are not available, and productive investment is not liquid or collateralizable. Thus, only liquid assets such as fiat money and government bonds may serve as precautionary measures or provide self-insurance for unobservable idiosyncratic shocks. We follow Dutta and Kapur by introducing government-issued assets as intergenerational allocation devices and thereby allow middle-aged consumers to insure themselves against idiosyncratic shocks by holding such assets from when they are young. In particular, young investors are allowed to hold $m$ units of government-issued assets, and middle-aged consumers can keep unspent government-issued assets until they reach old age.

Throughout this paper, we focus on the steady-state equilibrium and ignore the transition to equilibrium. In addition, the welfare is measured in terms of the expected utility of a new born in the steady state. This is mainly because we are interested in the long-run welfare consequences of fiscal and monetary policies.\(^7\)

These government-issued assets can be interpreted as government bonds with zero real rates of interest that do not depreciate and when there are no aggregate shocks. At the same time, they can also be interpreted as fiat money. In particular, given depreciation at the rate of $\pi$, it is reasonable to interpret the government-issued assets as fiat money under nominal price changes that depreciate in value by the single-period rate of inflation $\pi$. Because we consider monetary and fiscal policy, we adopt the fiat-money interpretation. However, fiat money is equivalent to zero-real-interest-rate government bonds if nominal prices are constant over time.

Now, we explain decisions about consumption profiles. We assume that middle-aged and old, but not young, individuals consume. There is no bequest motive. To make

\(^7\) For our theoretical and numerical analyses, there is also a technical reason for focusing on long-run welfare comparisons. Without any aggregate shock, in our model, equilibrium returns on government bonds (equilibrium inflation rates in a monetary economy) are constant in the steady state. Indeed, the proofs of theoretical propositions and the numerical procedures rely heavily on the constancy of equilibrium interest rates (inflation rates). However, when we analyze a transition path toward a steady state, it is impossible to prove most theoretical propositions under time-varying interest rates (inflation rates). It is also extremely difficult to conduct numerical calculations in this case.
a current portfolio choice between liquid assets $m$ and illiquid assets $I$, and to make a future consumption plan, a young investor maximizes the following lifetime expected utility function, which incorporates logarithmic preferences:

$$U \equiv \sum_{i=L,h} \Pr(y_i) \{u(c_1(y_i)) + u(c_2(y_i))\}$$

$$= \frac{1}{2} (\ln c_1(y_l) + \ln c_2(y_l)) + \frac{1}{2} (\ln c_1(y_h) + \ln c_2(y_h)), \quad (1)$$

where $c_1(y)$ is the consumption level of the middle-aged individual contingent on a realization of middle-aged income $y$, and $c_2(y)$ is the consumption level of an old person.

To see the basic properties of this model under perfect insurance, let us consider the special case in which expected middle-aged income is unity and the initial endowment $y_0$ satisfies the following:

$$(1 + r) y_0 = \frac{1}{2} (y_h + y_l) = 1. \quad (2)$$

The steady-state first-best allocation, in which idiosyncratic shocks are insured perfectly, implies that a young investor achieves $c_1(y_h) = c_2(y_h) = c_1(y_l) = c_2(y_l) = 1$ by allocating his or her entire endowment to irreversible investment ($I = y_0$). The corresponding welfare level $W$ is equal to zero. Conversely, if a young investor allocates a part of his or her initial endowment to liquid assets for precautionary purposes in the steady-state equilibrium, the level of investment is less than $1/(1 + r)$ ($I < y_0$). In this case, consumers may experience a reduction in long-run welfare through having sacrificed productive opportunities. Thus, an increase in liquid assets may lower welfare. We further examine this simple case (2) extensively in the numerical examples of Section 3.

In this framework, Dutta and Kapur (1998) and Saito and Takeda (2006) both attempt to identify the financial instruments that might be used to improve welfare. Dutta and Kapur introduce several types of financial intermediation to address the problem of insufficient productive investment, whereas Saito and Takeda examine the possibility of using dynamic insurance contracts with incentive compatibility constraints to improve welfare. Unlike these authors, we explore the fiscal and monetary policies that might enhance welfare in this context. We examine how a government should finance subsidies to those who can commit to irreversible investments (young investors). In particular, we investigate lump-sum subsidies that are financed by lump-sum taxes, consumption taxes, and inflation taxes. Note that income taxes cannot be used as an instrument because income is assumed
to be unobservable.

2.2. **Cases without subsidies: a frame of reference** Before undertaking positive and normative evaluations of various combinations of taxes and subsidies, we first investigate the case in which there are no changes in nominal prices as a frame of reference.

A young investor allocates a portfolio between fiat money \( m \) (measured in real terms) and irreversible investments \( I \), given the budget constraint \( I + m = y_0 \) with \( 0 \leq m \leq y_0 \). When middle-aged income \( y \) is realized, a middle-aged consumer chooses \( c_1(y) \) subject to the liquidity constraint \( c_1(y) \leq m + y \). An old consumer’s consumption \( c_2(y) \) is financed by the return on investment and unspent money balances; that is, \( c_2(y) = (1 + r)(y_0 - m) + m + y - c_1 \).

The utility maximization problem is solved backwards. Given \( m \) and a realization of \( y \in \{ y_l, y_h \} \), a middle-aged consumer chooses \( c_1(m, y) \). Let \( \lambda(m, y) \) be the Lagrangian multiplier associated with the liquidity constraint. If liquidity constraints for middle-aged consumers are binding, we obtain \( c_1(m, y) = m + y, c_2(m, y) = (1 + r)(y_0 - m) \), and \( \lambda(m, y) = \frac{1}{m+y} - \frac{1}{(1+r)(y_0-m)} \). Given that \( c_1 \) (or \( c_2 \)) is increasing (or decreasing) in \( m \), holding more money helps to smooth the consumption profile. In this case, we have \( m < \frac{(1+r)y_0 - y}{2+r} \). However, if liquidity constraints for middle-aged consumers are not binding, we have \( c_1(m, y) = c_2(m, y) = \frac{1}{2}((1 + r)(y_0 - m) + m + y) \) and \( \lambda(m, y) = 0 \). The consumption profile for middle-aged and old consumers is flat because the rate of time preference is zero. In this case, the inequality \( m \geq \frac{(1+r)y_0 - y}{2+r} \) should be satisfied.

The multiplier \( \lambda \) serves as the shadow price of the liquidity constraint; that is, consumers are willing to pay a price of \( \lambda \) to relax the liquidity constraint by one unit. A higher value of \( \lambda \) implies that the liquidity constraint is more binding. With binding constraints, either greater money holding \( m \) or lower opportunity costs \( r \) lower \( \lambda \). With no binding liquidity constraints, we have \( \lambda = 0 \).

Substituting \( c_1(m, y_l), c_2(m, y_l), c_1(m, y_h) \), and \( c_2(m, y_h) \) into (1) yields an indirect utility function conditional on money holding \( m \). By maximizing their indirect utility, young investors choose their optimal money holdings \( m^* \). Following Saito and Takeda (2006), we focus on a monetary economy, in which money demand is positive and liquidity constraints are binding only for low-income earners. The following proposition states the conditions for such an economy and demonstrates the properties of \( m^* \).

**Proposition 1**: Assume that \( y_l < (1 + r)y_0 < y_h \).
1. Liquidity constraints are binding only for low-income earners.

2. Money demand $m^*$ is positive when 
   \[ 0 < r < \frac{(y_0 - y_l)(y_0 + y_h)}{(3y_l - y_0)y_0}. \]
   
3. If $m^*$ is positive, then it is increasing in $y_h$, and decreasing in both $y_l$ and $r$.

**Proof.** See Appendix A.

From the first statement, it is clear that assumption (2) used extensively in Section 3 is a sufficient condition for liquidity constraints to be binding only for low-income earners. The second statement implies that the more volatile are income processes, the more likely it is that consumers with negative income shocks are liquidity constrained and that money demand is positive. This is because higher income volatility expands the range of returns on investment $r$ for any positive money demand, because either higher $y_h$ or lower $y_l$ raises
   \[ \frac{(y_0 - y_l)(y_0 + y_h)}{(3y_l - y_0)y_0}. \]
   From the third statement, optimal money demand is increased by either increased income volatility or a reduced opportunity cost of money holding.

Because there are only two markets (for consumption goods and fiat money) in the cross-sectional allocation, Walras’ Law implies that we can focus on the following goods-market clearing condition:

\[
I(m^*) + \frac{1}{2} [(c_1(m^*, y_h) + c_1(m^*, y_l)) + (c_2(m^*, y_h) + c_2(m^*, y_l))] = y_0 + \frac{1}{2}(y_h + y_l) + (1 + r)I(m^*). \tag{3}
\]

The right-hand (left-hand) side of equation (3) represents aggregate supply (demand). Note that this market clearing condition is satisfied if $m^*$ is determined by the first-order condition, equation (4) in Appendix A, because (3) is implicitly used to derive (4).

### 2.3. Fiscal policy for financing lump-sum subsidies to investors

In this subsection, we examine the positive implications of using two types of fiscal instrument to finance lump-sum subsidies to young investors, namely (1) lump-sum taxes and (2) consumption taxes. As mentioned earlier, income taxes are not available as an instrument under our assumption that income is unobservable. We assume that there is no inflation, $(\pi = 0)$. Then, real money holdings ($m$) may also be interpreted as government bonds with zero real rates of interest.

First, we consider the case of lump-sum taxes. The total amount of lump-sum taxes is denoted by $\tau_0$. It is assumed that consumers are identifiable in terms of their cohort so that different tax burdens can be imposed according to age. The allocation of lump-sum
taxes between middle-aged and old consumers is parameterized by $0 \leq k \leq 1$; that is, tax burdens are imposed only on old consumers when $k = 0$, only on middle-aged consumers when $k = 1$, and are broadly based when $k = 0.5$.

A lump-sum tax with $k \neq 0.5$ is an unrealistic policy instrument in terms of implementability, but it may serve as a thought experiment to illuminate the role of lump-sum taxes as a redistribution device in incomplete markets. Only a broadly based lump-sum tax ($k = 0.5$) may be a reasonable and realistic candidate for welfare comparisons.

Let $s$ denote the size of the lump-sum subsidy to young investors. Then, the budget constraints of households are $c_1 \leq m + y - k\tau_0$ and $c_2 = (1 + r)(y_0 + s - m) + m + y - c_1 - \tau_0$. The first panel of Table 1 summarizes optimal consumption and the Lagrange multiplier $\lambda$, given money demand $m$, when liquidity constraints are binding only for low-income earners. As suggested in Table 1, given $m$, when the tax burden shifts from middle-aged ($k = 1$) to old consumers ($k = 0$), liquidity constraints are relaxed and thus $\lambda$ is reduced. As shown later, this analytical property plays an important role in determining the welfare effects of lump-sum taxation.

Given the balanced government budget constraint $s = \tau_0$, optimal money demand $m^*$ is positive when there is lump-sum taxation, if money demand is positive when there are no subsidies.\footnote{This follows from equations (4) and (5) in Appendix A.}

Next, we consider a policy combination in which lump-sum subsidies to young investors are financed by broadly based consumption taxes. The budget constraints of consumers are $(1 + \tau_1)c_1 \leq m + y$ and $(1 + \tau_1)c_2 = (1 + r)(y_0 + s - m) + m + y - (1 + \tau_1)c_1$, where $\tau_1$ is the rate of consumption tax. The second panel of Table 1 summarizes optimal consumption and the Lagrange multiplier, given money demand $m$, when liquidity constraints are binding only for low-income earners.

Provided that money demand is positive when there are no subsidies, optimal money demand $m^*$ is positive when there are subsidies financed by consumption taxes.\footnote{This follows from equations (4) and (6) in Appendix A.}

The following proposition demonstrates the global properties of the regime with lump-sum subsidies financed by lump-sum and consumption taxes.

**Proposition 2**: Suppose that money demand is positive and that liquidity constraints are binding only for low-income earners. Then:

---

\[\text{This follows from equations (4) and (5) in Appendix A.}\]

\[\text{This follows from equations (4) and (6) in Appendix A.}\]
1. Both money demand \( m^* \) and productive investment \( I \) increase with the size of the lump-sum subsidies \( s \) except when there are lump-sum taxes with \( k = 1 \).

2. In the case of lump-sum taxes, a decrease in \( k \) (a shift of the tax burden from middle-aged consumers to old consumers) leads to a decrease in \( m^* \) and an increase in \( I \).

3. When there are consumption taxes, if \( 0 < r < 1 \), money demand (productive investment) is greater (smaller) than when lump-sum taxes are imposed on only old consumers, and is smaller (greater) than when lump-sum taxes are imposed on only middle-aged consumers, for a given level of subsidy.

**Proof.** See Appendix A.

Proposition 1 shows that money demand is positive and that liquidity constraints are binding only for low-income earners under fairly general assumptions when there is no subsidy \( (s = 0) \). Thus, the four claims of Proposition 2 apply around \( s = 0 \). Proposition 2 shows that the four claims hold for a large \( s > 0 \), provided that money demand is positive and liquidity constraints are binding only for low-income earners.\(^{10}\) This remark applies to all propositions in this paper unless otherwise stated.

There is an interesting feature of this case. When lump-sum taxes are imposed only on middle-aged consumers \( (k = 1) \), the optimal amount of fiat money (liquid assets) increases by the same amount as do the lump-sum subsidies, whereas productive investment is independent of the size of the subsidies. In other words, the imposition of lump-sum taxation on middle-aged consumers only has no effect on consumption plans or lifetime expected utility. However, this neutrality is a rather trivial consequence of liquidity constraints being irrelevant between young and middle-aged consumers by construction; it is assumed that the young do not consume.

For lump-sum taxes with \( 0 \leq k < 1 \), both investment \( I \) and money demand are monotonically increasing in the size of the lump-sum subsidies \( s \). In Subsection 2.5, we show that with relaxed liquidity constraints for low-income earners because of greater money

---

\(^{10}\) Because the exact conditions for positive money demand and binding liquidity constraints for low-income earners vary with tax and subsidy schemes and become complicated, we do not present them.
holding, the promotion of productive investment financed by lump-sum taxes improves welfare. Moreover, we demonstrate that the greater is the burden of lump-sum taxation on old consumers, the higher is welfare.

In standard dynamic models with incomplete insurance, a stronger demand for precautionary savings instruments (money in this case) indicates that consumers are more exposed to uninsured shocks. This implies that consumers suffer a larger reduction in welfare. In this regard, the welfare effects of imposing lump-sum taxation on only old consumers would be expected to dominate those of broadly based consumption taxation. However, as we demonstrate in Subsection 2.5, this is not necessarily the case in our model of reversible and uncollateralizable investment with no insurance. This is because consumption taxation is preferable in relation to risk sharing between high- and low-income earners because of its redistributive nature.

2.4. Monetary policy for financing lump-sum subsidies to investors

Instead of financing through taxation, we now consider money financing for lump-sum subsidies to young investors. Specifically, the government redistributes seigniorage revenues from inflation (inflation taxes) to young investors in a lump-sum manner. This is the case discussed extensively by Dutta and Kapur (1998) and Saito and Takeda (2006) in different contexts.

The budget constraints of consumers are $c_1 \leq (1-\pi)m + y$ and $c_2 = (1+r)(y_0 + s - m) + (1-\pi)[(1-\pi)m + y - c_1]$, where $\pi$ denotes the rate of inflation. The third panel of Table 1 reports optimal consumption and the Lagrange multiplier, given real money holdings $m$, when liquidity constraints are binding for low-income earners only. According to this table, larger money holdings and a lower rate of inflation help to smooth the consumption profile of low-income earners. In addition, a higher rate of inflation distorts the intertemporal consumption allocation even for high-income earners who are free of liquidity constraints.

The equilibrium inflation rate $\pi$ is determined by the goods-market equilibrium condition (3). It is clear that inflation is zero ($\pi = 0$) when seigniorage revenues and thus subsidies are zero ($s = 0$). To satisfy the government budget constraint, nominal money supply $M$ must increase by $M_{t+1} - M_t = sP_{t+1} = s(1 + \pi)P_t$ ($P$ is a nominal price) in order to maintain the steady-state monetary equilibrium. We use numerical examples in the next section to obtain the global properties of money finances.
2.5. **Normative implications of lump-sum subsidies**  So far, we have investigated the positive aspects of the macroeconomic policy of providing lump-sum subsidies financed by either lump-sum taxes, consumption taxes, or inflation taxes. In this subsection, we consider the normative implications of these policy combinations by evaluating lifetime expected utility in the steady-state equilibrium.

The following proposition summarizes the welfare implications of lump-sum subsidies financed by different financial instruments.

**Proposition 3:** Suppose that money demand is positive and that liquidity constraints are binding only for low-income earners.

1. An increase in the level of subsidies $s$ raises lifetime expected utility either when $s$ is financed by lump-sum taxes and $0 \leq k < 1$, or when $s$ is financed by consumption taxes and $0 < r < 2$.

2. Zero inflation is suboptimal. That is, when there are inflation taxes, an increase in $s$ from $s = 0$ improves welfare.

3. In the case of lump-sum taxes, a decrease in $k$ (which represents a shift in the lump-sum tax burden from middle-aged to old consumers) improves welfare.

4. If $0 < r < 2$ and $\frac{1}{4} \left( \sqrt{1 + k + \frac{1}{r}} - 1 \right) (y_h - y_l) > \left( 1 - \frac{1}{2r + 1} \right) (y_0 + y_l)$, then the welfare effect of broadly based consumption taxes exceeds that of lump-sum taxes in the neighborhood of $s = 0$.

**Proof.** See Appendix A.

The first statement of the above proposition implies that, when there are consumption taxes and lump-sum taxes with $k \neq 1$, increasing subsidies to young investors improves welfare by increasing both irreversible investment and money demand; an increase in investment directly expands long-run output, whereas an increase in money demand relaxes the liquidity constraint for middle-aged consumers.

The second statement indicates that an increase in the inflation rate triggered by raising seigniorage revenues improves lifetime expected utility by expanding irreversible investment at the expense of holding money,\(^{11}\) provided that subsidies are positive but small.

---

\(^{11}\) As Dutta and Kapur (1998) point out, this property corresponds to the Tobin effect.
However, we should be careful in inferring the global properties of money financing from this local result. An even higher rate of inflation (which increases the cost of holding money) generates two types of distortions. As suggested in the third panel of Table 1, a higher rate of inflation tightens liquidity constraints and thus reduces middle-aged consumption for low-income earners. In addition, money financing distorts the intertemporal allocation even for high-income earners, who are free from liquidity constraints. Thus, the cost of holding money may dominate the benefit from inflation taxes when subsidies are high. The numerical examples in Section 3 confirm this conjecture.

The third statement shows that a shift in the tax burden from middle-aged to old consumers has redistributive effects on welfare. As the first panel of Table 1 shows, a decrease in $k$ relaxes the liquidity constraint for middle-aged earners, and helps to smooth their consumption substantially. In the sense that middle-aged low-income consumers bear the highest tax burden relative to their incomes, a lump-sum tax on only old consumers is the most redistributive form of lump-sum taxation. This redistributive effect is welfare improving.

In the fourth statement, we provide the sufficient condition under which taxing consumption improves welfare by more than does imposing lump-sum taxes. We make two comments on \[ \left( \frac{1}{4} \left( \sqrt{1 + k + \frac{1}{r}} - 1 \right) (y_h - y_l) \right) > \left( 1 - \frac{k}{2 + r} \right) (y_0 + y_l). \]

First, as indicated in the proof of the proposition, this inequality is a sufficient condition. Even if this is not satisfied, it is possible to show numerically that consumption taxes dominate lump-sum taxes. Second, it is easy to show that the left-hand side of the condition is increasing in both $y_h$ and $k$ and decreasing in both $y_l$ and $r$, whereas the right-hand side is increasing in both $y_l$ and $r$. That is, with a higher lump-sum tax burden on middle-aged consumers ($k$), lower returns on investment ($r$), and a larger volatility of income ($v$, defined as $\frac{y_h - y_l}{2}$), this condition is likely to be satisfied. The latter property is illustrated by the heavily shaded area in Figure 1, where consumption taxes dominate lump-sum taxes on only old consumers ($k = 0$). This represents the most welfare-enhancing form of lump-sum taxation in the neighborhood of $s = 0$ under the assumption of equation (2).

The dominating welfare effect of consumption taxation over lump-sum taxation suggests which policies are desirable for promoting irreversible investment in the context of incomplete markets. As shown in the previous subsection (Proposition 3), imposing lump-sum taxes on only old consumers is preferable to consumption taxation for expanding long-
run consumption opportunities through productive investment. However, on redistributive grounds, consumption taxes are preferable to lump-sum taxes. The latter part of Proposition 3.4 below, although it relates to a local property around $s = 0$, suggests that the redistributive effects of consumption taxes dominate the consumption-increasing effects of lump-sum taxes under the above-mentioned condition.

Now, we compare welfare levels under tax financing and money financing. General conditions become extremely complicated when considering whether welfare is greater under money financing than under consumption tax financing, even when subsidies are minimal. In Appendix B, we identify these conditions in the neighborhood of $s = 0$ in special cases by making the simplifying assumption (2). We show that welfare is higher under money financing if: (i) returns on investment $r$ are close to zero, or (ii) income volatility (average absolute deviations of individual incomes) is not too large. These conditions are intuitive: lower returns on investment reduce the opportunity cost of holding money, and the redistributive aspect of consumption taxes has a more limited role when income volatility is smaller.

Conversely, if consumers face rather volatile shocks to incomes when returns on investment $r$ are high, it is possible for consumption taxes to dominate inflation taxes in the neighborhood of zero subsidies. This is illustrated by the lightly shaded area of Figure 2 where consumption taxes dominate inflation taxes in the neighborhood of zero subsidies under the assumption of equation (2).

2.6. **Subsidies proportional to productive investment** Before presenting numerical examples, we briefly consider the case in which a government provides young investors with subsidies proportional to the amount of their irreversible investment. Let $\rho$ be a proportional rate of investment subsidies. Then, the budget constraint for a young investor is $(1 - \rho)I + m = y_0$. Given that the amount of subsidies is now $\frac{\rho}{1 - \rho}(y_0 - m)$, we can derive the optimal holding of fiat money (liquid assets) and consumption plans in the same manner as before.

Because of analytical difficulties in the case of proportional subsidies, we resort to numerical examples in the next section. However, the following basic property of proportional subsidies is easy to understand. Proportional subsidies differ from lump-sum subsidies because they directly increase the opportunity cost of holding money and distort the portfolio allocation between irreversible investment and liquid assets. Such a distortion may reduce
lifetime expected utility. This additional distortion makes analytical examination difficult.

3. **Numerical Examples** In the previous section, we examined the positive and normative implications of policy combinations of subsidies to young investors and taxes on middle-aged and/or old consumers in incomplete markets. In this section, we evaluate these policy combinations quantitatively. Our numerical examples are based on Japanese macroeconomic data.

3.1. **Parameter setting** Under our three-period overlapping generations framework, one period is assumed to correspond to a period of 15 to 20 years. We assume that average middle-aged income is normalized to unity \((0.5(y_h + y_l) = 1)\). Given this normalization, we specify the values of the following parameters: \(v\) (the standard deviation of middle-aged uninsured income shocks); \(r\) (the rate of return on irreversible investment); and \(y_0\) (the young's endowment).

First, we consider the standard deviation of middle-aged uninsured income \(v\) \((= y_h - 1 = 1 - y_l)\). Because the young’s endowment \(y_0\) is assumed to be constant in our framework, \(v\) can be interpreted as the difference in the standard deviations of young and middle-aged consumers. Based on the empirical results of Ohtake and Saito (1998), we set \(v\) to 0.5. By using Japanese household microdata, Ohtake and Saito show that income inequality in terms of the log-variance increases by between 0.2 and 0.3 from the age of 25 to the age of 55. Thus, we set \(v\) to 0.5 \((= \sqrt{(0.3 + 0.2)/2})\).

Second, the value of the parameter for the rate of return on irreversible investment \(r\) is set to 0.2. Because the rate of time preference is zero in our model, we may interpret \(r\) as the difference between the rate of return on physical investment and the rate of return on risk-free assets for a period of 30 to 40 years. Thus, we adopt a relatively conservative premium for \(r\).

Third, we make the simplifying assumption that \((1 + r)y_0 = 1\), which implies that the level of welfare based on the first-best allocation is zero. Because middle-aged consumers’ incomes are 20% higher than those of young consumers on average, this assumption reflects the fact that average income increases with age.

Given the above set of parameters, Proposition 1 implies that, in the case of no subsidies \((s = 0)\), liquidity constraints are binding only for low-income earners, and money demand is positive. As is shown below, the model yields reasonable predictions in terms of capital income shares, consumption inequality, and the ratio of liquid to illiquid assets.
The predicted capital income share, \((1 + r)I/\left[ y_0 + 0.5(y_h + y_l) + (1 + r)I \right] \), is 0.315. This is similar to the actual value. According to Hayashi and Prescott (2004), the sample average of the capital income share was 0.362 between 1984 and 1989. Predicted economy-wide consumption inequality amounts to 0.1012 in terms of the variance of logarithmic consumption. Ohtake and Saito (1998) report economy-wide log-consumption variances ranging from 0.20 to 0.24 for 1979, 1984, and 1989. Our prediction implies that half of consumption inequality can be explained by uninsured income shocks; the other half may be accounted for by business cycle shocks, demographic shocks, and aggregate shocks that are disproportionate among consumers.

The predicted ratio of liquid to illiquid assets \((m/I)\) is 0.186. For a corresponding ratio in the real economy, we use the Japanese national accounts to calculate the ratio of M2 plus government bonds to nonfinancial assets. This ratio averaged 0.36 between 1994 and 2003. The predicted and observed ratios may differ for two reasons. First, the numerator of the average ratio in the data may be overestimated. Because some Japanese government bonds are issued to finance public construction, they are partially backed assets. Second, the denominator may be underestimated because the nonfinancial assets reported in the national accounts do not cover nonphysical irreversible investment such as human capital.

In the following subsections, based on our chosen parameters, we investigate the effects of policy combinations on long-run welfare, money, and investment. We consider cases in which subsidies are not only small but are also large. We examine small-scale lump-sum subsidies, then small-scale subsidies proportional to the amount of investment, and then large-scale lump-sum and proportional subsidies.

### 3.2. Small-scale lump-sum subsidies to investors

We first analyze cases in which there are relatively small-scale lump-sum subsidies to young investors. Figures 3, 4, and 5 plot the levels of welfare (lifetime expected utility), money, and investment, respectively, against the level of the lump-sum subsidies for each policy combination. Because \(0.5(y_h + y_l) = 1\), the level of lump-sum subsidies \(s\) can be expressed as a percentage of average middle-aged income.

Lump-sum subsidies have no effect on lifetime expected utility or investment when there are lump-sum taxes only on middle-aged consumers \((k = 1)\). As discussed in Subsection 2.3, even though young investors receive subsidies, they use all of them to purchase government bonds in preparation for paying next period’s taxes.
As shown in Propositions 3.1 and 2.1, when \( k \) is below unity (so more weight is assigned to old consumers), lump-sum subsidies to young investors promote welfare, irreversible investment, and money demand. Increasing investment helps to expand long-run production opportunities. Increasing money demand relaxes liquidity constraints for low-income consumers and helps to smooth their consumption. As implied by Proposition 3.3, the closer \( k \) is to zero, the more significantly do the promotion of productive investment and the consumption-smoothing effect jointly enhance welfare. Consequently, taxing only old consumers \((k = 0)\) dominates any other type of lump-sum taxation.

As shown in Proposition 3.1, in the case of broadly based consumption taxes, welfare is monotonically increasing with the level of lump-sum subsidies. As in the case of lump-sum taxes, lump-sum subsidies to young investors promote not only irreversible investment but also money demand (see Proposition 2.1). However, an interesting difference from the case of lump-sum taxes is that welfare is higher under broadly based consumption taxes than under lump-sum taxation with \( k = 0.5 \). This means that solving only underinvestment problems does not necessarily result in welfare improvement. As discussed in detail below, broadly based consumption taxation generates greater redistribution between high- and low-income earners than does lump-sum taxation. Given this redistributive effect, consumption taxation substantially improves welfare.\(^{12}\) However, as \( k \) approaches zero, the relative welfare advantage of broadly based consumption taxes gradually diminishes (see Proposition 3.3); in that case, the effects of consumption taxation are almost identical to the effects of the lump-sum taxation of only old consumers.\(^{13}\)

As Figure 3 shows, financing by seigniorage revenues enhances welfare as does financing through either consumption taxation or the lump-sum taxation of only old consumers when lump-sum subsidies increase marginally from zero (see Proposition 3.2). Seigniorage-revenue financing, which reduces money holdings (Figure 4), is the most effective of the five methods of promoting physical investment (Figure 5). These properties are interpreted as follows. An increase in the inflation rate triggered by an increase in seigniorage revenues

---

\(^{12}\) Additional numerical exercises, not reported in this paper, yield the following two results. First, financing through broadly based consumption taxes has the same welfare effect as financing through lump-sum taxes with \( k = 0.09 \). Second, if volatility \( v \) is set to 0.6, welfare is higher under broadly based consumption taxation than under lump-sum taxation of only old consumers. The second result implies that when middle-aged consumers face more volatile income shocks, the redistributive effect on welfare becomes more significant than the effect of promoting physical investment.

\(^{13}\) The sufficient condition in Proposition 3.4 is not satisfied.
leads to an increase in the opportunity cost of holding money but also raises productive investment and expands aggregate output at the expense of money demand.

Welfare levels deteriorate quickly as the level of the subsidies increases, although the amount of investment monotonically increases with lump-sum subsidies. As Figure 3 shows, the optimal level of seigniorage is low (at about 1.0% of average middle-aged income).

A major reason for the hump-shaped welfare curve under money financing is that the benefits of solving underinvestment in irreversible capital are quickly dominated by the cost of money holdings. Larger seigniorage revenues increase inflation, or the cost of holding money. Costly money holding not only makes it difficult for young investors to hold money for precautionary reasons but also makes it difficult for high-income earners to transfer resources from middle-age to old age, even though high-income earners are free of liquidity constraints. In this regard, the difference in money financing between small and large subsidies is analogous to the difference between broadly based consumption taxes and lump-sum taxes.

By using Tables 2, 3, and 4, we can consider the redistributive aspects of the above cases. Table 2 evaluates the consumption profiles at five values of lump-sum subsidies for the five policy combinations. With both lump-sum and consumption taxes, high-income earners can achieve perfect consumption smoothing. Values of $k$ closer to zero under lump-sum taxation make it easier for low-income earners to consume on relatively smooth paths at higher levels. When there is consumption taxation, low-income earners can enjoy relatively smooth consumption at even higher levels. Under money financing, not only low-income earners but also high-income earners fail to smooth consumption intertemporally.

Table 3 reports the effective tax burden rate for each policy combination. The tax burden rate is defined as the percentage of total income paid in tax, with income consisting of endowments and savings balances held in terms of fiat money or government bonds. As this table shows, high-income (low-income) earners have a heavier (lighter) tax burden under consumption taxes than under lump-sum taxes with $k = 0.5$. In this respect, broadly based consumption taxes are more redistributive relative to broadly based lump-sum taxes. In addition, lump-sum taxes on only middle-aged consumers are the least redistributive in the sense that low-income earners bear the highest burden, whereas broadly based con-

\[^{14}\text{In the case of consumption taxes, the tax burden rate for consumers is } \tau/(\tau + 1), \text{ except for middle-aged high-income consumers, who save part of their income.}\]
sumption taxes are the most redistributive in that low-income earners bear the lowest burden.

Table 4 reports the Lagrange multiplier associated with the budget constraint of low-income middle-aged consumers. As discussed in Section 2, the higher is the multiplier, the more severe are the liquidity constraints. Given the level of subsidies, the value of the multiplier is highest under money financing, and lowest under consumption taxation. In other words, liquidity constraints for low-income earners are tightest under money financing and are loosest under consumption taxation. Relaxing their liquidity constraints helps low-income earners to smooth their consumption. These results also corroborate the relative advantage of consumption taxes in terms of self-insurance capabilities.

3.3. Proportional investment subsidies We now consider small-scale subsidies proportional to the amount of irreversible investment. Figure 6 plots the level of welfare (lifetime expected utility) against the level of the subsidies.\textsuperscript{15} As shown in this figure, the welfare effect of proportional subsidies is similar to that of lump-sum subsidies. Although not reported here, investment and money demand also exhibit similar patterns to those under lump-sum subsidies. One noticeable difference, however, is that unlike lump-sum subsidies, proportional subsidies reduce welfare when there are lump-sum taxes on only middle-aged consumers. In addition, according to Figure 7, which compares the two types of subsidies when they are financed by consumption taxes, welfare is lower under proportional subsidies (the solid lines) than under lump-sum subsidies (the dotted line).

A major reason for these numerical results is that the allocation between irreversible investment and liquid assets (money) is distorted by proportional subsidies. Proportional subsidies promote irreversible investment only at the expense of money demand, thereby making liquidity constraints more binding for middle-aged low-income earners. In particular, lifetime utility deteriorates substantially when proportional subsidies are financed either by seigniorage revenues or by lump-sum taxes on only middle-aged consumers. As already discussed, under both types of financing, the capability of middle-aged consumers to self-insure against income risks is substantially undermined not only by proportional subsidies but also by less redistributive taxes.

\textsuperscript{15} The level of the subsidies is calculated from \( \frac{\rho}{1 - \rho} (y_0 - m) \), where \( \rho \) is the rate of subsidies.
3.4. **Large-scale subsidies** We now analyze large-scale subsidies. As already shown, the cost of money-financed subsidies quickly outweighs their benefits, even for small-scale subsidies. However, welfare increases with the level of the subsidies when the subsidies are small and are financed by taxation (except in the case of lump-sum taxes on only middle-aged consumers). Hence, we focus on large-scale subsidies financed either by broadly based lump-sum taxes or by broadly based consumption taxes.

We first consider the case of lump-sum subsidies for productive investment, which corresponds Proposition 3.1. As shown in Figure 8, lifetime welfare is monotonically increasing even for large subsidies under both types of financing, whereas financing by consumption taxes is more desirable than financing by lump-sum taxes. We make two comments on this monotonicity property. First, the property holds as long as middle-aged low-income earners are subject to liquidity constraints and money demand is positive. Second, the welfare level under $s = 40\%$ remains below the welfare level under the first-best allocation (zero in our framework).

Turning to proportional subsidies (Figure 9), we find that financing by consumption taxes is again preferable to financing by lump-sum taxes for large-scale subsidies. However, marginal welfare is diminishing under both types of financing. The larger are the subsidies, the greater is the cost of distorting allocations relative to the benefit of promoting irreversible investment. In particular, as the hump-shaped curve indicates, welfare under lump-sum taxation begins to deteriorate when $s = 10\%$. Less redistributive taxes further hamper the capability of self-insuring against income risks.

4. **Concluding Remarks** In this paper, we explored the extent to which fiscal and monetary instruments help to enhance long-run welfare when financial markets are incomplete in the sense that productive investment is irreversible and uncollateralizable, there is no insurance for idiosyncratic shocks to income, and only government-issued bonds provide self-insurance. Unlike in dynamic models with reversible physical capital, in our model, an increase in precautionary savings in the form of holding liquid bonds reduces, rather than increases, irreversible productive investment. We demonstrated that subsidies to promote productive but irreversible investment should be financed without reducing the capabilities of consumers to self-insure against idiosyncratic shocks. When income is not perfectly observable, which renders income taxes ineffective, lump-sum subsidies financed by consumption taxes are preferred to fixed and/or proportional investment subsidies financed by
either large-scale seigniorage revenues or lump-sum taxes. According to our model, combining lump-sum subsidies and consumption taxes is more redistributive and thus more consumption smoothing than is using other sets of instruments.

The advantage of consumption taxes arises because the unobservable components of income shocks are reflected in the level of consumption. As a result, imposing taxes proportional to consumption levels could generate redistributive effects from consumers experiencing positive shocks to those experiencing negative shocks. This policy implication also applies when there are unobservable preference shocks. This is because preference shocks have direct impacts on individual consumption.

As has been emphasized throughout the paper, the irreversibility of productive investment plays an essential role in determining both positive and normative implications. However, in our model, productive capital is completely irreversible before maturity occurs, and only government-issued assets serve as liquid assets. An important extension of this model would be the introduction of partially irreversible privately issued assets through the incorporation of a set of illiquid assets with different maturities. Under interaction with macroeconomic policies in such a dynamic context, term structures may emerge not only for interest rates but also for liquidity premiums or wedges in the returns between liquid and illiquid assets, whereas the allocation of productive capital between different maturities is determined endogenously.

We focused on steady-state welfare comparisons partly because of our interest in long-run policy consequences, and partly because of analytical simplicity. Nevertheless, analyzing the transition towards a steady state is important, particularly in overlapping generations models such as ours. Thus, we identify transition analysis as an important task for future research.

Appendix A: Proofs of the Propositions

Proof of Proposition 1 1. Let indirect utility conditional on real money holdings m be $V(m)$:

$$V(m) \equiv \max_{c_1, c_2, I} \left[ \frac{1}{2} \{ \ln c_1(m, y_l) + \ln c_2(m, y_l) \} + \frac{1}{2} \{ \ln c_1(m, y_h) + \ln c_2(m, y_h) \} \right]$$
Because of our simple framework, it is straightforward to show that

\[
V(m) = \begin{cases} 
V_1(m), & \text{if } m < \frac{(1+r)y_0-y_h}{2+r}, \\
V_2(m), & \text{if } \frac{(1+r)y_0-y_h}{2+r} \leq m \leq \frac{(1+r)y_0-y_h}{2+r}, \\
V_3(m), & \text{if } m > \frac{(1+r)y_0-y_h}{2+r}, 
\end{cases}
\]

where

\[
V_1(m) = \frac{1}{2} \ln(m + y_l) + \frac{1}{2} \ln(m + y_h) + \ln(1 + r)(y_0 - m),
\]

\[
V_2(m) = \frac{1}{2} \ln(m + y_l) + \frac{1}{2} \ln(1 + r)(y_0 - m) + \ln \left[ \frac{1}{2} ((1 + r)(y_0 - m) + m + y_h) \right],
\]

\[
V_3(m) = \ln \left[ \frac{1}{2} ((1 + r)(y_0 - m) + m + y_l) \right] + \ln \left[ \frac{1}{2} ((1 + r)(y_0 - m) + m + y_h) \right].
\]

Note that if \( m < \frac{(1+r)y_0-y_h}{2+r} \left( m < \frac{(1+r)y_0-y_h}{2+r} \right) \), then liquidity constraints are binding for high-income (low-income) earners.

The assumption \( y_l < (1+r)y_0 < y_h \) implies \( \frac{(1+r)y_0-y_h}{2+r} < 0 \) and \( \frac{(1+r)y_0-y_h}{2+r} > 0 \). The condition \( \frac{(1+r)y_0-y_h}{2+r} < 0 \) implies that the range of \( V_1(m) \) never overlaps \( \{ m : 0 \leq m \leq y_0 \} \), while the condition \( \frac{(1+r)y_0-y_h}{2+r} > 0 \) indicates that the ranges of \( V_2(m) \) and \( \{ m : 0 \leq m \leq y_0 \} \) intersect. Furthermore, \( V_3(m) \) is monotonically decreasing in \( m \). Therefore, the optimal level of money \( m^* \) (the solution of \( \max V(m) \text{ s.t. } 0 \leq m \leq y_0 \)) lies in the range of \( V_2(m) \). This implies that liquidity constraints are binding only for low-income consumers.

2. For \( m^* \) to be positive, it is sufficient that \( \left. \frac{\partial V_2}{\partial m} \right|_{m=0} = \frac{0.5}{y_l} - \frac{0.5}{y_h} - \frac{r}{(1+r)y_0+y_h} > 0 \). Algebraic manipulation reveals that \( r < \frac{(y_0-y_l)(y_0+y_h)}{(3y_h-y_0)y_0} \) is necessary and sufficient.

3. If money demand is positive so that liquidity constraints are binding only for low-income earners, we have \( V(m) = V_2(m) \). Consequently, the optimal holdings of \( m^* \), if they are positive, must satisfy \( \left. \frac{\partial V_2}{\partial m} \right|_{m=m^*} = 0 \), or

\[
\frac{1}{m^* + y_l} = \frac{1}{y_0 - m^*} + \frac{2r}{(1+r)(y_0 - m^*) + m^* + y_h}. \tag{4}
\]

A fall in \( y_l \) raises the left-hand side of equation (4), while an increase in \( y_h \) or a fall in \( r \) reduces the right-hand side. Hence, the optimal level of money is increased by either a rise in income volatility or by fall in the opportunity cost of holding money.

**Proof of Proposition 2**  To prove all statements of the proposition, we analyze lump-sum taxes and consumption taxes separately.
In the lump-sum taxes case, we prove the first and second statements about money and the first and second statements about investment. Optimal money demand $m^*$ satisfies the following condition:

$$\frac{1}{m^* + y_t - k\tau_0} = \frac{1 + r}{(1 + r)(y_0 + s - m^*)} + \frac{2r}{(1 + r)(y_0 + s - m^*) + m^* + y_h - \tau_0}. \tag{5}$$

Thus, the first and second statements about money demand clearly follows from the above equation with $\tau_0 = s$.

We prove the first and second statements about investment. Substituting $m' = m^* - s$ and $\tau_0 = s$ into equation (5) yields

$$\frac{1}{m' + y_t + (1 - k)s} = \frac{1}{y_0 - m' - \frac{1 + k}{1 + r}s} + \frac{2r}{(1 + r)(y_0 - m') + m' + y_h}. $$

The left-hand side of the above equation is decreasing in $m'$, while its right-hand side is increasing in $m'$ when $r > 0$. Either an increase in $s$ or a decrease in $k$ lowers the left-hand side, but raises the right-hand side. Given positive money demand, $m'$ must fall to equate both sides. Thus, $I = y_0 - m'$ is increasing in $s$ and decreasing in $k$.

In the case of consumptions taxes, we prove the first and third statements about money and the first and third statements about investment. Optimal money demand $m^*$ satisfies the following condition:

$$\frac{1}{m^* + y_t} = \frac{1}{y_0 + s - m^*} + \frac{2r}{(1 + r)(y_0 + s - m^*) + m^* + y_h}. \tag{6}$$

Thus, the first statement about money demand clearly follows from the above equation.

To prove the third statement about money demand, we show that money demand under consumption taxation is equal to that under lump-sum taxation for some $0 < k < 1$.

Let $m^*$ and $m^0$ be the optimal money demand under consumption taxes and lump-sum taxes with $k = 0$, respectively. Then, $m^*$ satisfies equation (6) or

$$\frac{1}{m^* + y_t} = \frac{1}{y_0 + s - m^*} + \frac{2r}{(1 + r)(y_0 + s - m^*) + m^* + y_h},$$

while $m^0$ satisfies equation (5) with $s = \tau_0$ and $k = 0$, or

$$\frac{1}{m^0 + y_t} = \frac{1}{y_0 + s - m^0 - \frac{s}{1 + r}} + \frac{2r}{(1 + r)(y_0 + s - m^0) + m^0 + y_h - s}. $$
The left-hand (right-hand) sides of these equations are decreasing (increasing) in money demand. Under the assumption that $m^* = m^0$, the left-hand sides of both equations are equal to each other, while the right-hand of the first equation is smaller than that of the second. Hence, $m^* > m^0$.

Let $m^1$ be optimal money demand under lump-sum taxation with $k = 1$. Then, $m^1$ satisfies equation (5) with $s = \tau_0$ and $k = 1$, or

$$
\frac{1}{m^1 + y_l - s} - \frac{2r}{(1 + r)(y_0 + s - m^1) + m^1 + y_h - s} = \frac{1}{y_0 + s - m^1}.
$$

Equation (6) becomes

$$
\frac{1}{m^* + y_l} - \frac{2r}{(1 + r)(y_0 + s - m^*) + m^* + y_h} = \frac{1}{y_0 + s - m^*}.
$$

The left-hand (right-hand) sides of the above equations are decreasing (increasing) in money demand. Because the right-hand sides of these equations are equal to each other when $m^* = m^1$, and because the left-hand side of the second equation is smaller than that of the first (given that $0 < r < 1$ by assumption), then $c_1(m, y_l) < c_1(m, y_h)$ (or $m + y_l - s < 0.5\{(1 + r)(y_0 + s - m) + m + y_h - s\}$) for $s (\geq 0)$ and $m$. Hence, $m^* < m^1$.

The second statement of this proposition implies that there exists some $0 < k < 1$ such that optimal money demand under lump-sum taxation is equal to that under consumption taxation. The third statement about money holdings follows from $0 < k < 1$.

Because $I = y_0 - s + m$, and given the second statement, the statements about investment are true.

**Proof of Proposition 3** 1. Given optimal money demand $m^*$, the welfare level $W_l$ ($W_c$) is derived by substituting the consumption plans reported in the first (second) panel of Table 1 into equation (1). We show that $\frac{dW_l}{ds} > 0$ and $\frac{dW_c}{ds} > 0$ below.

First, we prove that $\frac{dW_l}{ds} > 0$. Applying the envelope theorem to $W_l$ yields

$$
\frac{dW_l}{ds} = -\frac{0.5k}{m^* + y_l - ks} + \frac{0.5(r + k)}{(1 + r)(y_0 + s - m^*) - (1 - k)s} + \frac{r}{(1 + r)(y_0 + s - m^*) + m^* + y_h - s}. \tag{7}
$$

Substituting equation (5) with $s = \tau_0$ into equation (7) yields

$$
\frac{dW_l}{ds} = \frac{0.5(1 - k)}{m^* + y_l - ks} + \frac{0.5(k - 1)}{(1 + r)(y_0 - m^*) + (r + k)s} > 0.
$$

This is because $0 < k < 1$, $c_1(m^*, y_l) > 0$, and $c_2(m^*, y_l) > 0$. 

Next, we prove that \( \frac{dW^c}{ds} > 0 \). From the balanced-government-budget constraint, \( s = \frac{1}{2} \tau_1 (c_1(y_l, m^*) + c_2(y_l, m^*) + c_1(y_l, m^*) + c_2(y_l, m^*)) \), the equilibrium tax rate is determined by

\[
1 + \tau_1 = \frac{(1 + r)(y_0 + s - m^*) + m^* + \frac{1}{2}(y_l + y_h)}{(1 + r)(y_0 + s - m^*) + m^* + \frac{1}{2}(y_l + y_h) - s}.
\]

Substituting the above equation into \( W^c \) and applying the envelope theorem to \( W^c \) yields

\[
\frac{dW^c}{ds} = \frac{0.5}{y_0 + s - m^*} + \frac{1 + r}{(1 + r)(y_0 + s - m^*) + m^* + y_h} - \frac{2(1 + r)}{c} + \frac{2r}{c - s}.
\] (8)

where \( c = (1 + r)(y_0 + s - m^*) + m^* + 0.5(y_l + y_h) \). By substituting the first-order condition (6) into (8) and noting that \( \frac{2r}{c - s} > \frac{2r}{c} \), we obtain

\[
\frac{dW^c}{ds} > \frac{0.5}{m^* + y_l} + \frac{1}{(1 + r)(y_0 + s - m^*) + m^* + y_h} - \frac{2}{c}.
\] (9)

The right-hand side of (9) is decreasing in \( m^* \). In fact, the third term is decreasing in \( m^* \). The sum of the first and second terms is also decreasing, because the first derivative of both terms is as follows:

\[- \frac{0.5}{(m^* + y_l)^2} + \frac{r}{[(1 + r)(y_0 + s - m^*) + m^* + y_h]^2} < -0.5 \left[ \frac{1}{c_1(m^*, y_l)^2} - \frac{1}{c_1(m^*, y_h)^2} \right] < 0,\]

where \( c_1(m^*, y_l) = m^* + y_l \) and \( c_1(m^*, y_h) = 0.5((1 + r)(y_0 + s - m^*) + m^* + y_h) \). The first (second) inequality is established because \( 0 < r < 2 (c_1(m^*, y_l) < c_1(m^*, y_h)) \).

Thus, it is sufficient to show that the right-hand side of (9) is positive at \( m^* = \frac{(1 + r)(y_0 + s) - y_l}{2 + r} \), which represents the highest level of money demand that is consistent with liquidity constraints being binding for low-income earners. Because \( (1 + r)(y_0 + s - m^*) = m^* + y_l \) in this case, we find

\[
\frac{dW^c}{ds} > \frac{0.5}{m^* + y_l} + \frac{0.5}{m^* + 0.5(y_l + y_h)} - \frac{2}{m^* + y_l + m^* + 0.5(y_l + y_h)}.
\]

Denoting \( a \equiv m^* + y_l(> 0) \) and \( b \equiv m^* + 0.5(y_l + y_h)(> 0) \) yields

\[
\frac{dW^c}{ds} > \frac{0.5}{a} + \frac{0.5}{b} - \frac{2}{a + b} > \frac{0.5(b - a)^2}{ab(a + b)^2} > 0.
\]

2. This statement is essentially the same as the proposition in Saito and Takeda (2006). See the proof in their paper.
3. We prove that \( \frac{dW^l}{dk} < 0 \). Applying the envelope theorem to \( W^l \) leads to

\[
\frac{dW^l}{dk} = -\frac{0.5s}{m^* + y_l - ks} + \frac{0.5s}{(1+r)(y_0 + s - m^*) - (1-k)s}.
\]

Because \( c_1(m^*, y_l) < c_2(m^*, y_l) \) or \( m^* + y_l - ks < (1+r)(y_0 + s - m^*) - (1-k)s \), it follows that \( \frac{dW^l}{dk} < 0 \).

4. We prove that \( \left. \frac{dW^c}{ds} \right|_{s=0} - \left. \frac{dW^l}{ds} \right|_{s=0} > 0 \) below. If \( s = 0 \), then both types of taxation generate the same optimal money demand \( m^* \) and the same welfare level. Marginal welfare under consumption taxation at \( s = 0 \) is derived from equation (8), as follows:

\[
\left. \frac{dW^c}{ds} \right|_{s=0} = \frac{0.5}{y_0 - m^*} + \frac{1 + r}{(1+r)(y_0 - m^*) + m^* + y_h} - \frac{2}{c},
\]

where \( \tilde{c} = (1 + r)(y_0 - m^*) + m^* + 0.5(y_l + y_h) \). Marginal welfare under lump-sum taxation is derived from equation (7), as follows:

\[
\left. \frac{dW^l}{ds} \right|_{s=0} = -\frac{0.5k}{m^* + y_l} + \frac{0.5(r + k)}{(1+r)(y_0 - m^*) + m^* + y_h} - \frac{r}{y_0 - m^*}.
\]

Substituting (4) into \( \left. \frac{dW^c}{ds} \right|_{s=0} - \left. \frac{dW^l}{ds} \right|_{s=0} \), and some algebraic manipulation, yields

\[
\left. \frac{dW^c}{ds} \right|_{s=0} - \left. \frac{dW^l}{ds} \right|_{s=0} = \frac{1 + kr}{1 + r} \left[ \frac{0.5}{m^* + y_l} + \frac{1}{(1+r)(y_0 - m^*) + m^* + y_h} \right] - \frac{2}{c}.
\]

Because the term in the square brackets of (12) is equal to the first and second terms on the right-hand side of (9), the right-hand side of (12) is decreasing in \( m^* \).

Thus, as in the third part of this proposition, it is sufficient to show that the right-hand side of (12) is positive at \( m^* = \frac{(1+r)y_0 - y_l}{2 + r} \). By denoting \( a \equiv m^* + y_l = (1+r)(y_0 - m^*) > 0 \) and \( b \equiv \frac{1}{2} \{(1+r)(y_0 - m^*) + m^* + y_h\} = m^* + 0.5(y_l + y_h) > 0 \), we obtain

\[
\left. \frac{dW^c}{ds} \right|_{s=0} - \left. \frac{dW^l}{ds} \right|_{s=0} = \frac{0.5(1 + kr)}{1 + r} \left[ \frac{1}{a} + \frac{1}{b} \right] - \frac{2}{a + b} > 0.
\]

A sufficient condition for \( \left. \frac{dW^c}{ds} \right|_{s=0} - \left. \frac{dW^l}{ds} \right|_{s=0} > 0 \) is \( (b - a)^2 > 4rab \). Because \( a = \frac{1}{1 + r}(y_0 + y_l) \) and \( b = a + \frac{1}{2}(y_h - y_l) \), this condition can be rewritten as \( \frac{1}{4} \left( \sqrt{1 + k + \frac{1}{r} - 1} \right) (y_h - y_l) > \left( 1 - \frac{1}{2 + r} \right) (y_0 + y_l) \).
Appendix B: Welfare Comparisons under Inflation Taxes and Consumption Taxes

In this appendix, we discuss the conditions under which the level of welfare is higher under money financing than under consumption taxation in the neighborhood of \( s = 0 \). To make our analysis simpler and more intuitive, we make an additional simplifying assumption in addition to those of Propositions 1 through 3.

Suppose that \((1 + r)y_0 = 0.5(y_l + y_h) = 1\) holds. Then, we can show that welfare is higher under money financing than under consumption taxation in the neighborhood of \( s = 0 \) if either (i) the return on investment \( r \) is close to zero, or (ii) income volatility \( v \), defined below, is not too high. The following discussion clarifies the precise meanings of ‘close to’ and ‘not too high’ in this context.

For this purpose, we examine the sign of \( \frac{dW_p}{ds} \bigg|_{s=0} - \frac{dW_c}{ds} \bigg|_{s=0} \). With \((1 + r)y_0 = 0.5(y_l + y_h) = 1\) and \( v \equiv 1 - y_l = y_h - 1 \) together with equations (10) and (13), the condition for \( \frac{dW_p}{ds} \bigg|_{s=0} - \frac{dW_c}{ds} \bigg|_{s=0} > 0 \) can be rewritten as \( \phi(m, r, v) > 0 \), where

\[
\phi(m, r, v) = \frac{v + (6 + r)m}{2(2 - rm)} + \frac{1 - (2 + r)m}{2 + v - rm} - \frac{0.5m}{m + 1 - v} - 0.5.
\]

We investigate the conditions for \( \phi > 0 \) below.

First, we discuss the condition for \( r \) to satisfy \( \phi > 0 \). To prove statement (i) above, we show that if \( r \) is decreasing and close to zero (so that \( y_0 \) is close to unity), money demand increases to \( \frac{1}{2} \). Using \((1 + r)y_0 = 0.5(y_l + y_h) = 1\) enables us to write equation (4) as \( \psi(m, r) = 0 \), where

\[
\psi(m, r) = \frac{1}{m + 1 - v} - \frac{1}{1 + r - m} - \frac{2 + v}{r - m}. \tag{14}
\]

Because \( \psi(m, r) \) is increasing in \( r \) (decreasing in \( m \)), a decrease in \( r \) raises the optimal money demand \( m^* \) so that \( \psi(m^*, r) = 0 \). Furthermore, \( m^* = \frac{v}{2} \) solves \( \psi(m^*, 0) = 0 \), and \( \psi(m, r) \) is continuous in \( r \). Thus, \( \lim_{r \to 0} m^* = \frac{v}{2} \) holds. Given \( r = 0 \), it is straightforward to show that for
any income volatility $0 < v < 1$

$$\phi(0.5v, 0, v) = \frac{(1 - v)v^2}{(2 + v)(2 - v)} > 0.$$ 

Because $\phi$ is continuous in $r$, $\lim_{r \to 0} \phi > 0$. That is, if $r$ is close to zero, then $\phi > 0$.

Note that, when $r$ is positive, optimal money demand is less than $0.5v$, which implies that $\phi$ may be negative when $v$ is high. For example, when $r = 0.05$ and $v = 0.99$, $\phi < 0$ is confirmed numerically.

We next explore the range of $v$ satisfying $\phi > 0$. To prove statement (ii) above, we first show $\frac{\partial \phi}{\partial r} > 0$. Indeed, because $0 < m < 0.5v$, it follows that

$$\frac{\partial \phi}{\partial r} = \frac{2 + v + 6m}{2(2 - rm)^2} - \frac{1 + v + 2m}{(2 + v - rm)^2} > \frac{m(m - 0.5v)}{(2 + v - rm)^2} > 0.$$ 

For a sufficient condition, we determine the range of $v$ that satisfies $\phi > 0$ at $r = 0$. When $r = 0$,

$$\phi(m, 0, v) = \frac{v^2}{4(2 + v)} + 0.5m \left\{ \xi - \frac{1}{m + 2 - v} \right\},$$

where $\xi = \frac{3v + 2}{2 + v}$. Because $\frac{\partial \phi}{\partial m} = 0.5\xi - \frac{0.5(1 - v)}{(m + 1 - v)^2}$, $\phi$ takes the minimum value with $\hat{m} = \sqrt{yI/\xi - yI}$, and its value is calculated as $\phi(\hat{m}, 0, v) = \frac{v^2}{4(2 + v)} - 0.5 \left( 1 - \sqrt{\xi(1 - v)} \right)^2$. Accordingly, $\phi(\hat{m}, 0, v) > 0$ is equivalent to $\zeta(v) > 0$, where

$$\zeta(v) = v - \sqrt{2(2 + v)} + \sqrt{2(3v + 2)(1 - v)}.$$ 

It is easy to show that $\zeta(0) = 0$, $\zeta'(0) > 0$, and $\zeta(1) = -1$. These equations imply that $\zeta > 0$ when $v$ is not too high.

Note that the above condition is sufficient. For example, $\zeta(0.6) > 0$ and $\zeta(0.7) < 0$ are confirmed numerically when $r$ is close to zero. However, inflation taxation still dominates consumption taxation when $v = 0.7$ and $r = 0.3$.

In our numerical exercises presented in Section 3, we consider the case in which $v$ is relatively high but $r$ is small, so that a sufficiently high positive level of money demand emerges. In this case, money financing is the best policy instrument in the neighborhood of $s = 0$.

REFERENCES


Table 1: Analytical forms of consumption profiles and Lagrange multipliers in the case of lump-sum subsidies

<table>
<thead>
<tr>
<th>Consumption Type</th>
<th>Lump-sum tax</th>
<th>Consumption tax</th>
<th>Seigniorage Revenues</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1(m, y_l)$</td>
<td>$m + y_l - k\tau_0$</td>
<td>$c_1(m, y_l)$</td>
<td>$c_1(m, y_h)$</td>
</tr>
<tr>
<td>$c_2(m, y_l)$</td>
<td>$(1 + r)(y_0 + s - m) - (1 - k)\tau_0$</td>
<td>$0.5((1 + r)(y_0 + s - m) + m + y - \tau_0)$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\lambda(m, y_l)$</td>
<td>$\frac{m + y_l - k\tau_0}{(1 + r)(y_0 + s - m) - (1 - k)\tau_0}$</td>
<td>$c_1(m, y_h)$</td>
<td>$c_1(m, y_h)$</td>
</tr>
<tr>
<td>$c_1(m, y_h)$</td>
<td>$0.5((1 + r)(y_0 + s - m) + m + y - \tau_0)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$c_2(m, y_h)$</td>
<td>$(1 + r)(y_0 + s - m) + m + y - \tau_0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda(m, y_h)$</td>
<td>$1 - \pi$</td>
<td></td>
<td>$0$</td>
</tr>
</tbody>
</table>

32
Table 2: Consumption profiles in the case of lump-sum subsidies

<table>
<thead>
<tr>
<th>Consumption Tax</th>
<th>s = 0.0%</th>
<th>s = 0.5%</th>
<th>s = 1.0%</th>
<th>s = 1.5%</th>
</tr>
</thead>
<tbody>
<tr>
<td>lump-sum tax (k = 0)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( c_1(m, y_l) )</td>
<td>0.631</td>
<td>0.631</td>
<td>0.632</td>
<td>0.632</td>
</tr>
<tr>
<td>( c_2(m, y_l) )</td>
<td>0.843</td>
<td>0.844</td>
<td>0.844</td>
<td>0.845</td>
</tr>
<tr>
<td>( c_1(m, y_h) = c_2(m, y_h) )</td>
<td>1.237</td>
<td>1.237</td>
<td>1.238</td>
<td>1.238</td>
</tr>
<tr>
<td>lump-sum tax (k = 0.5)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( c_1(m, y_l) )</td>
<td>0.631</td>
<td>0.631</td>
<td>0.631</td>
<td>0.631</td>
</tr>
<tr>
<td>( c_2(m, y_l) )</td>
<td>0.843</td>
<td>0.843</td>
<td>0.844</td>
<td>0.844</td>
</tr>
<tr>
<td>( c_1(m, y_h) = c_2(m, y_h) )</td>
<td>1.237</td>
<td>1.237</td>
<td>1.237</td>
<td>1.237</td>
</tr>
<tr>
<td>lump-sum tax (k = 1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( c_1(m, y_l) )</td>
<td>0.631</td>
<td>0.631</td>
<td>0.631</td>
<td>0.631</td>
</tr>
<tr>
<td>( c_2(m, y_l) )</td>
<td>0.843</td>
<td>0.843</td>
<td>0.843</td>
<td>0.843</td>
</tr>
<tr>
<td>( c_1(m, y_h) = c_2(m, y_h) )</td>
<td>1.237</td>
<td>1.237</td>
<td>1.237</td>
<td>1.237</td>
</tr>
<tr>
<td>consumption tax</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( c_1(m, y_l) )</td>
<td>0.631</td>
<td>0.632</td>
<td>0.632</td>
<td>0.633</td>
</tr>
<tr>
<td>( c_2(m, y_l) )</td>
<td>0.843</td>
<td>0.844</td>
<td>0.845</td>
<td>0.846</td>
</tr>
<tr>
<td>( c_1(m, y_h) = c_2(m, y_h) )</td>
<td>1.237</td>
<td>1.237</td>
<td>1.236</td>
<td>1.236</td>
</tr>
<tr>
<td>seigniorage revenues</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( c_1(m, y_l) )</td>
<td>0.631</td>
<td>0.620</td>
<td>0.608</td>
<td>0.591</td>
</tr>
<tr>
<td>( c_2(m, y_l) )</td>
<td>0.843</td>
<td>0.859</td>
<td>0.878</td>
<td>0.902</td>
</tr>
<tr>
<td>( c_1(m, y_h) )</td>
<td>1.237</td>
<td>1.247</td>
<td>1.259</td>
<td>1.275</td>
</tr>
<tr>
<td>( c_2(m, y_h) )</td>
<td>1.237</td>
<td>1.227</td>
<td>1.215</td>
<td>1.200</td>
</tr>
</tbody>
</table>
Table 3: Effective rates of tax burdens in the case of tax-financing and lump-sum subsidies (unit: %)

<table>
<thead>
<tr>
<th>financing instrument</th>
<th>s = 0.0%</th>
<th>s = 0.5%</th>
<th>s = 1.0%</th>
<th>s = 1.5%</th>
</tr>
</thead>
<tbody>
<tr>
<td>lump-sum tax (k = 0)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>middle-aged low-income</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>old low-income</td>
<td>0.00</td>
<td>0.59</td>
<td>1.17</td>
<td>1.75</td>
</tr>
<tr>
<td>middle-aged high-income</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>old high-income</td>
<td>0.00</td>
<td>0.40</td>
<td>0.80</td>
<td>1.20</td>
</tr>
<tr>
<td>lump-sum tax (k = 0.5)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>middle-aged low-income</td>
<td>0.00</td>
<td>0.39</td>
<td>0.79</td>
<td>1.17</td>
</tr>
<tr>
<td>old low-income</td>
<td>0.00</td>
<td>0.30</td>
<td>0.59</td>
<td>0.88</td>
</tr>
<tr>
<td>middle-aged high-income</td>
<td>0.00</td>
<td>0.15</td>
<td>0.31</td>
<td>0.46</td>
</tr>
<tr>
<td>old high-income</td>
<td>0.00</td>
<td>0.20</td>
<td>0.40</td>
<td>0.60</td>
</tr>
<tr>
<td>lump-sum tax (k = 1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>middle-aged low-income</td>
<td>0.00</td>
<td>0.79</td>
<td>1.56</td>
<td>2.32</td>
</tr>
<tr>
<td>old low-income</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>middle-aged high-income</td>
<td>0.00</td>
<td>0.31</td>
<td>0.61</td>
<td>0.91</td>
</tr>
<tr>
<td>old high-income</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>consumption tax</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>middle-aged low-income</td>
<td>0.00</td>
<td>0.25</td>
<td>0.50</td>
<td>0.75</td>
</tr>
<tr>
<td>old low-income</td>
<td>0.00</td>
<td>0.25</td>
<td>0.50</td>
<td>0.75</td>
</tr>
<tr>
<td>middle-aged high-income</td>
<td>0.00</td>
<td>0.19</td>
<td>0.38</td>
<td>0.57</td>
</tr>
<tr>
<td>old high-income</td>
<td>0.00</td>
<td>0.25</td>
<td>0.50</td>
<td>0.75</td>
</tr>
</tbody>
</table>

(1) The effective rate of tax burdens is defined as the percentage ratio of tax payment relative to total income including endowment as well as saving balances held in terms of fiat money or government bonds.

Table 4: Lagrange multipliers in the case of lump-sum subsidies

<table>
<thead>
<tr>
<th>financing instrument</th>
<th>s = 0.0%</th>
<th>s = 0.5%</th>
<th>s = 1.0%</th>
<th>s = 1.5%</th>
</tr>
</thead>
<tbody>
<tr>
<td>lump-sum tax (k = 0)</td>
<td>0.3989</td>
<td>0.3987</td>
<td>0.3985</td>
<td>0.3983</td>
</tr>
<tr>
<td>lump-sum tax (k = 0.5)</td>
<td>0.3989</td>
<td>0.3988</td>
<td>0.3987</td>
<td>0.3986</td>
</tr>
<tr>
<td>lump-sum tax (k = 1)</td>
<td>0.3989</td>
<td>0.3989</td>
<td>0.3989</td>
<td>0.3989</td>
</tr>
<tr>
<td>consumption tax</td>
<td>0.3989</td>
<td>0.3977</td>
<td>0.3964</td>
<td>0.3952</td>
</tr>
<tr>
<td>seigniorage revenues</td>
<td>0.3989</td>
<td>0.4669</td>
<td>0.5463</td>
<td>0.6485</td>
</tr>
</tbody>
</table>
Figure 1: Welfare comparison between the consumption taxation and the lump-sum taxation in the neighborhood of zero subsidies

(1) A non-shaded area represents the case with zero money demand. A heavily shaded area depicts the case where the consumption taxation dominates the lump-sum taxation on only old consumers in the neighborhood of zero subsidies, while a lightly shaded area draws the reverse case.

(2) An income volatility $v$ is defined as $\frac{1}{2}(y_h - y_l)$.

(3) It is assumed that $(1 + r)y_0 = \frac{1}{2}(y_h + y_l) = 1$. 


Figure 2: Welfare comparison between the inflation taxation and the consumption taxation in the neighborhood of zero subsidies

(1) A non-shaded area represents the case with zero money demand. A lightly shaded area depicts the case where the consumption taxation dominates the inflation taxation in the neighborhood of zero subsidies, while a heavily shaded area draws the reverse case.

(2) An income volatility $v$ is defined as $\frac{1}{2}(y_h - y_l)$.

(3) It is assumed that $(1 + r)y_0 = \frac{1}{2}(y_h + y_l) = 1.$
Figure 3: Welfare comparison in the case of lump-sum subsidies

(1) The size of lump-sum subsidies is expressed as a percentage value relative to the average of middle-aged income.

Figure 4: Money demand comparison in the case of lump-sum subsidies

(1) The size of lump-sum subsidies is expressed as a percentage value relative to the average of middle-aged income.
Figure 5: Physical investment comparison in the case of lump-sum subsidies

(1) The size of lump-sum subsidies is expressed as a percentage value relative to the average of middle-aged income.

Figure 6: Welfare comparison in the case of subsidies proportional to investment

(1) The size of lump-sum subsidies is expressed as a percentage value relative to the average of middle-aged income.
Figure 7: Welfare comparison in the case of consumption taxes

Figure 8: Welfare comparison between consumption taxes and lump-sum taxes \((k = 0.5)\) in the case of large-scale lump-sum subsidies

(1) The size of lump-sum subsidies is expressed as a percentage value relative to the average of middle-aged income.

(1) The size of lump-sum subsidies is expressed as a percentage value relative to the average of middle-aged income.
Figure 9: Welfare comparison between consumption taxes and lump-sum taxes ($k = 0.5$) in the case of *large-scale* subsidies proportional subsidies

(1) The size of lump-sum subsidies is expressed as a percentage value relative to the average of middle-aged income.