Condorcet Jury Theorem or Rational Ignorance*

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Abstract

We analyze a symmetric model of an election in which voters are uncertain which of two alternatives is desirable for them. Each voter must incur some cost to acquire information about the alternatives. We show that by focusing on *unbiased* voting strategies, general symmetric signal structures can be degenerated to a two-signals model. In addition, we show that for any sequence of unbiased voting equilibria, if the second-order derivative of the information cost function at no information is zero, then this probability converges to one, that is, the Condorcet Jury Theorem is valid. Otherwise, the probability converges to some value less than one, that is, the "rational ignorance" hypothesis is valid.

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1 Introduction

Can elections aggregate private information in a large society? This is a central question concerning elections. Apropos of this question, the Condorcet Jury Theorem (hereafter CJT) provided a powerful justification for elections under the majority-voting rule. CJT asserts that the probability of making an appropriate decision will converge to one as the number of voters grows (see Condorcet, 1785, or Black, 1958). There is a great deal of literature advocating CJT.[1] Nevertheless, CJT may not be valid if each voter voluntarily acquires costly information. Increasing the number of voters reduces incentives for information acquisition because the probability of affecting the outcome becomes small.

In his recent pioneering work, Martinelli (2004) showed that CJT is valid under certain conditions even if information acquisition is costly. The main point of his paper is that "rationally ignorant" voters are consistent with a well-informed electorate. Formally, Martinelli (2004) considered the sequence of symmetric equilibria and showed that CJT is valid along this sequence if and only if the second derivative of the information cost function at no information is zero.

However, Martinelli (2004) employed a somewhat restrictive framework to reach this conclusion. Considering applications and the robustness of his results, there remain two problems to be solved: (i) there are only two signals in his model, and (ii) there still remains the possibility of asymmetric equilibria. We answer the following question in this paper: does Martinelli's result hold under general situations?

We study a symmetric model of an election in which voters are uncertain which of two alternatives is desirable for them. For each voter, the only information resource about the alternatives is a noisy signal, and he or she must incur some cost to reduce the noise. Typically, previous papers have dealt with simple signal structures, e.g., there are only two signals. In contrast, we consider general symmetric signal structures. We allow a wide variety of signal structures such that signals perturbed by a noise term are distributed continuously.

Our first result is that if all voters adopt *unbiased* voting strategies, such that each vote is symmetric between two alternatives, then the problem can be reduced to one parameter, i.e., the probability that the desirable alternative in reality looks more plausible to the signal receiver (Theorem 1). As a result, a symmetric signal structure can be degenerated to the simple one in which each voter can predict the desirable alternative with his or her own probability q. Thus, employing the simple signal structure is justified.

Our second result concerns the asymptotic property of the election outcome. Strategic voting models generally possess a multiplicity of equilibria. However, we show that for any sequence of unbiased voting equilibria, the probability of electing the desirable alternative converges to a unique value as the number of voters grows (Theorem 2). As a corollary of this result, combined with Theorem 2 in Martinelli (2004), we can calculate the asymptotic probability of the electoral outcome. Consequently, we show that if the

[1] See, e.g., Austen-Smith and Banks, 1996, Feddersen and Pesendorfer, 1997, and Berend and Paroush, 1998.

second-order derivative of the information cost function is zero at no information, then for any sequence of unbiased voting equilibria, the probability of electing the desirable alternative converges to one, that is, CJT is valid. Otherwise, this probability converges to p less than one, that is, CJT is not valid.

Although many studies on CJT have been conducted, most assumed that each voter's competence is given exogenously. This assumption ignores salient behavior of voters: for the purpose of voting for a desirable alternative, each voter would be willing to acquire information about the alternatives before the election. Taking such behavior into consideration, we suppose costly information acquisition. More precisely, for each voter, the only information resource about the alternatives is a noisy signal, and the voter must incur some cost to reduce the noise. Such endogenous information acquisition exerts a significant influence on electoral outcomes (see Martinelli, 2004, and Mukhopadhaya, 2003).

The "rational ignorance" hypothesis asserted by Downs (1957) is of significance to information aggregation under costly information acquisition. That is, voters will have no incentive to acquire information about the alternatives before voting because it is extremely rare that each vote becomes pivotal in a large election. Therefore, it seems at a glance that elections lead to a poorer decision in a large society, that is, CJT is not valid.

This inference, however, is not necessarily true. Even if the amount of information acquired by each voter is small, the amount of aggregated information can be large enough to result in the correct decision. Martinelli (2004) showed this, confining the problem to the sequence of symmetric equilibria. We show that his result is true under any general signal structure for any sequence of equilibria.

The remainder of the paper is organized as follows. Section 2 examines the model, and Section 3 derives two main results. Section 4 considers heterogeneous cost functions. Section 5 states the conclusions. All formal proofs are provided in the Appendix.

2 The Model

We analyze a symmetric election with two alternatives, L and R. It is assumed that either of the alternatives could be commonly desirable. There are $2n+1$ voters, indexed by i. A voter's payoff depends on the chosen alternative $d \in \{L, R\}$, the state $z \in \{z_L, z_R\}$, and the quality of information acquired by the voter before the election $x \in [0, \bar{x}] = X$. Acquiring information of quality x has a cost given by $C(x)$, so the payoff to a voter can be written as:

$$u(d, z) - C(x).$$

We assume that $x = 0$ is equivalent to acquiring no information, and that acquired information becomes more precise as x increases. In addition, we assume that $C(0) = 0$, $C(\bar{x}) = \infty$, and $C(\cdot)$ is increasing. L

(R) is the desirable alternative in state z_L (z_R):

$$u(d,z) = \begin{cases} 1 & \text{if } (d,z) = (L, z_L) \text{ or } (R, z_R), \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

At the beginning of time, nature selects the state with equal probability: $\Pr(z_L) = \Pr(z_R) = 1/2$. Voters are uncertain about the realization of the state and they can only observe independent noisy signals. After the realization of the state, each voter i must determine the quality of his or her information x_i. After deciding on x_i, i receives a private signal $s \in S$, which is independent among voters. The distribution of signals depends on both the quality of information x and the state z. Formally, let S be the signal space, $\mathcal{S}$ be the σ-algebra of subsets of S, and $\{\mu_d^x\}$ be the family of probability measures on $\mathcal{S}$. Given (x, z_d), the distribution of signals follows μ_d^x. We restrict our attention to symmetric signal structures.

Assumption 1

There exists a family of one-to-one transformations $\{\tau_x\}_{x \in X}$ on S such that for any $s \in S$,

$$\tau_x(\tau_x(s)) = s,$$

and for any $S' \in \mathcal{S}$,

$$\mu_L^x(S') = \mu_R^x(\tau_x(S')).$$

We call $\tau_x(s)$ the conjugate signal of s with respect to (w.r.t.) x. Obviously, the conjugate signal of $\tau_x(s)$ w.r.t. x is s. Intuitively, the probability of receiving s under z_L is equal to that of receiving $\tau_x(s)$ under z_R. Note that under Assumption 1, we obtain $\Pr\left(z_L \middle| s, x\right) = \Pr\left(z_R \middle| \tau_x(s), x\right)$ by Bayes' rule. That is, if z_L is plausible under s, then z_R is plausible under $\tau_x(s)$. We give two examples of symmetric signal structures.

Example 1 (Two signals: Martinelli, 2004)
$S = \{s_L, s_R\}$, $X = [0, 1/2]$, and the probability of receiving signal s_d in state z_d is given by $1/2 + x$. $\tau_x(s_L) = s_R$ and $\tau_x(s_R) = s_L$ for all x. We can regard s_d as the correct signal in the state z_d. This signal structure is the simplest one that describes the situation considered by Condorcet. $\|$

Example 2 (Normal noise: Kitahara and Sekiguchi, 2005a)
$S = (-\infty, \infty)$, $X = [0, \infty]$ and

$$s(x) = \begin{cases} -x + \epsilon & \text{if } z = z_L, \text{ and} \\ x + \epsilon & \text{if } z = z_R, \end{cases}$$

where $\epsilon \sim N(0,1)$. Obviously, $\tau_x(s) = -s$ for all x. $\|$

The election takes place after voters receive their signals. A voter can either vote for L or for R (that is, there is no abstention.) The alternative gaining most votes is chosen.

A strategy for a voter i is a tuple (x_i, v_i), where x_i specifies a quality of information, and a measurable function $v_i : S \to [0,1]$ specifies the probability of voting for L after receiving a signal. As we assume no abstention, the probability of voting for R given s is $1 - v_i(s)$.

A strategy profile $(\mathbf{x}, \mathbf{v})$ is a *voting equilibrium* if it is a Nash equilibrium. We restrict our attention to some subset of voting equilibria in which each vote is unbiased between two alternatives. We define an *unbiased strategy* as follows:

Definition 1 *A strategy (x_i, v_i) is unbiased if $v_i(s) = 1 - v(\tau_{x_i}(s))$ for all $s \in S$.*

Namely, for any signal, the probability of voting for L given this signal is equal to the probability of voting for R given the conjugate one. An *unbiased voting equilibrium* is a voting equilibrium in which all voters adopt unbiased strategies. We restrict our attention to unbiased voting equilibria for the purpose of excluding trivial equilibria such as, for example, all voters voting for L regardless of their signals. The following two points should be noted. First, we do not assume that strategies are symmetric among voters, that is, voters may adopt different unbiased strategies. Second, we do not exclude deviation to a biased strategy, e.g., $v_i(s) = 1$ for all s.

Remarks:

We refer to our election model as symmetric in the following sense:

(i) Each alternative is desirable with equal probability ex ante: $\Pr(z_L) = \Pr(z_R)$.

(ii) The payoff from the desirable alternative is independent of the names of the alternatives: Eq. (1).

(iii) The signal structure is symmetric: Assumption 1.

(iv) In equilibria, each vote is symmetric between the alternatives: Definition 1.

3 Main Results

3.1 Degeneration to two signals

In this subsection, we show that if all voters adopt unbiased strategies, then any symmetric signal structure can be degenerated to the two-signals case, as in Example 1. We begin with some preliminary definitions.

First, we define a family of partitions of S w.r.t. x as follows:

$$\begin{aligned} S^L(x) &= \{s \in S | \Pr(z_L|s,x) > \Pr(z_R|s,x)\}, \\ S^M(x) &= \{s \in S | \Pr(z_L|s,x) = \Pr(z_R|s,x)\}, \\ S^R(x) &= \{s \in S | \Pr(z_L|s,x) < \Pr(z_R|s,x)\}, \end{aligned}$$

where $x \in X$. Obviously, for any $x \in X$, $\{S^L(x), S^M(x), S^R(x)\}$ constitutes a partition of S and each element is measurable. $S^L(x)$ ($S^R(x)$) is the set of signals under which z_L (z_R) is more plausible when the quality of information is x, and $S^M(x)$ is the set of signals under which both states are similarly plausible. Notice that, from Assumption 1 and Bayes' rule, $s \in S^L(x)$ if and only if $\tau_x(s) \in S^R(x)$, and $s \in S^M(x)$ if and only if $\tau_x(s) \in S^M(x)$. As we will show below, in any unbiased voting equilibria each voter who chooses x and receives $s \in S^L(x)$ ($S^R(x)$) votes for L (R).

Next, we define accuracy of information $q : X \to [1/2, 1]$ as:

$$
\begin{aligned}
q(x) &= \mu_L^x\big(S^L(x)\big) + \frac{1}{2}\mu_L^x\big(S^M(x)\big) \\
&= \mu_R^x\big(S^R(x)\big) + \frac{1}{2}\mu_R^x\big(S^M(x)\big).
\end{aligned}
$$

By Assumption 1, $q(x)$ is well defined. In other words, accuracy $q(x)$ is the probability that the desirable alternative in reality looks more plausible for a voter with his or her quality of information x. Thus, if a voter with x votes sincerely, i.e., as if his or her vote alone determines the outcome, then the probability that he or she votes for the desirable alternative is equal to $q(x)$. Note that $q(0) = 1/2$ because we assumed that there is no information when $x = 0$. For Example 1, $q(x) = x + 1/2$. For Example 2, $q(x) = \Phi(x)$, where Φ is the cumulative distribution function of $N(0, 1)$. After this, $q(x)$ plays a key part in our analysis. For simplicity, we exclude a redundant x, namely, no two qualities have the same q. In addition, we assume that any $q(x) \in [1/2, 1]$ can be attained by choosing x appropriately.

Assumption 2

$q(\cdot)$ is a bijection, and hence, $\tilde{C} = C \circ q^{-1}$ is well defined.

$\tilde{C}(q)$ is the cost of information when an accuracy is q. In the remainder of the paper, we consider $\tilde{C}$ instead of C. From Assumption 2, selecting $x \in X$ is equivalent to selecting $q \in [1/2, 1]$ for a voter. Note that $\tilde{C}(1/2) = C(0) = 0$. For Examples 1 and 2, $\tilde{C}$ is well defined.

To simplify notations, let y_i for $i = 1, ..., 2n + 1$ be the random variable such that:

$$
y_i = \begin{cases} 1 & \text{if } i \text{ votes for the desirable alternative,} \\ 0 & \text{otherwise.} \end{cases}
$$

The only time i can influence the outcome of the election is if i's vote is pivotal, i.e., $\sum_{j \neq i} y_j = n$. A voter will vote for L if the expected payoff from voting for L, conditional on his or her vote being pivotal, is greater than the expected payoff from voting for R.

If voter i adopts an unbiased strategy, then $\Pr(y_i = 1|z_L) = \Pr(y_i = 1|z_R)$. Hence, in unbiased voting equilibria, the probability of i's vote being pivotal is the same as in both states, i.e., $\Pr(\sum_{j \neq i} y_j = n|z_L) = \Pr(\sum_{j \neq i} y_j = n|z_R)$, and the votes of others provide no information for i. Therefore, each voter votes informatively, that is, he or she votes for the alternative that is more likely to be desirable given his or her signal. Thus, we obtain the following lemma:

Lemma 1 *Suppose that* $(\mathbf{x}^*, \mathbf{v}^*)$ *is an unbiased voting equilibrium. Then,* v_i^* *satisfies the following for all* i*,*

$$v_i^*(s) = \begin{cases} 1 & \text{if } s \in S^L(x_i^*) \quad (a.s.), \\ 0 & \text{if } s \in S^R(x_i^*) \quad (a.s.), \end{cases} \tag{2}$$

$$E\Big[v_i^*(s)\Big| s \in S^M(x_i^*)\Big] = \frac{1}{2}. \tag{3}$$

Moreover, $\Pr\Big(y_i = 1\Big|(x_i^*, v_i^*)\Big) = q(x_i^*)$.

Eq. (3) is immediately obtained from the definition of the unbiased strategy. This lemma shows that in unbiased voting equilibria, all voters vote sincerely, and that the probability that i votes for the desirable alternative is exactly equal to $q(x_i^*)$. Hence, the equilibrium conditions are described by the accuracy of information $q(x_i^*)$. Before we state this formally, let $y_i(q)$ be the random variable that is independently distributed as:

$$y_i(q) = \begin{cases} 1 & \text{with probability } q \\ 0 & \text{with probability } 1 - q. \end{cases}$$

Theorem 1

For any unbiased voting equilibrium $(\mathbf{x}^*, \mathbf{v}^*)$*, for all* i*:*

$$q(x_i^*) \in \arg \max_{q \in [1/2, 1]} \left\{ q \Pr\Big(\sum_{j \neq i} y_j\big(q(x_j^*)\big) = n\Big) - \tilde{C}(q) \right\}. \tag{4}$$

Conversely, if x_i^* *satisfies Eq. (4) for all* i*, then there exists an unbiased voting equilibrium, in which each voter* i *chooses* x_i^**. The probability of electing the desirable alternative is equal to* $\Pr(\sum_i y_i(q(x_i^*)) > n)$*.*

This theorem reveals that for any symmetric signal structure, the optimization problem for voters is equivalent to the maximization problem w.r.t. $q(x_i)$. Therefore, any symmetric signal structure can be degenerated to the two-signals case, as in Example 1.

Next, we consider the existence of symmetric unbiased voting equilibria. To guarantee this existence, we make the following regular assumption in the remainder of the paper:

Assumption 3

$\tilde{C}$ *is strictly increasing, strictly convex, and twice continuously differentiable.*

For Examples 1 and 2, if C is strictly increasing, strictly convex, and twice continuously differentiable, then so is $\tilde{C}$ ($\tilde{C}$ is given by $C(q - 1/2)$ and $C(\Phi^{-1}(q))$, respectively).

Corollary 1 *Assume* $\tilde{C}'(1/2) = 0$*. Then, for an arbitrary* n*, there exists a symmetric unbiased voting equilibrium, in which all voters choose the same quality* $\bar{x}_n$*, which solves:*

$$\binom{2n}{n}(q_n(1 - q_n))^n = \tilde{C}'(q_n) \tag{5}$$

with $q_n = q(\bar{x}_n)$*.*

Martinelli also proved this statement for the two-signals case (Theorem 1 in Martinelli, 2004).

3.2 Asymptotic properties

We take an interest in the asymptotic property w.r.t. n. That is, we are interested in whether the probability of the desirable alternative being chosen by the election converges to one as n goes to infinity. Even for the two-signals case, the convergence property has been clarified only for the sequence of symmetric equilibria (see Martinelli (2004)). Nevertheless, the possibility of asymmetric equilibria remains. Generally, a multiplicity of equilibria is possible in strategic voting models. Can different performances occur by deliberately choosing asymmetric equilibria? The following theorem answers this question.

Theorem 2

Assume $\tilde{C}'(1/2) = 0$. Then, for any sequence of unbiased voting equilibria $\{(\mathbf{x}^n, \mathbf{v}^n)\}$,

(i) if $\tilde{C}''(1/2) > 0$, then for a sufficiently large n, $q(x_i^n) = q_n$ for all i, where q_n is the unique solution of Eq. (5);

(ii) if $\tilde{C}''(1/2) = 0$, then

$$\Pr\Big(\sum_i y_i\big(q(x_i^n)\big) > n\Big) \to 1 \text{ as } n \to \infty.$$

In other words, for any sequence of unbiased equilibria, the probability of the desirable alternative winning the election converges to that of the symmetric equilibrium sequence defined in Corollary 1. Combining this theorem with Martinelli's result, the convergence property of *all* equilibrium sequences is clarified. Theorem 2 in Martinelli (2004) asserted that if $\tilde{C}'(1/2) = 0$, and q_n satisfies (5) for all n, then $\Pr(\sum_i y_i(q_n) > n)$ approaches $\Phi(\delta)$, where δ solves:

$$\frac{\phi(\delta)}{\delta} = \frac{1}{4}\tilde{C}''(1/2), \tag{6}$$

where ϕ is the density function of $N(0,1)$. Hence, the following holds as a corollary of Theorem 2:

Corollary 2 *Assume $\tilde{C}'(0) = 0$. Then, for any sequence of unbiased voting equilibria, the probability of electing the desirable alternative converges to $\Phi(\delta)$, where δ is the unique solution of Eq. (6).*

We summarize the results obtained so far with the following remarks:

Remarks:

(i) If $\tilde{C}''(1/2) = 0$, then the probability of electing the desirable alternative approaches 1, that is, CJT is valid.

(ii) If $\tilde{C}''(1/2) \neq 0$, then the probability of electing the desirable alternative approaches $\Phi(\delta) < 1$, that is, the rational ignorance hypothesis is valid.

4 Heterogeneous Cost Functions

In this section, we consider cases where voters have heterogeneous cost functions. For simplicity, we assume that there are finite types of cost functions, $\{\tilde{C}_k\}_{k=1,\cdots,K}$. Let $m_{k,n}$ be the number of the voters who have $\tilde{C}_k$. We assume that for all k, the fraction of k-type voters converges to $\alpha_k > 0$, i.e., $m_{k,n}/(2n+1) \to \alpha_k$ as $n \to \infty$. In addition, we assume that for all k, $\tilde{C}_k$ satisfies Assumption 3.

Of course, Lemma 1 is valid even if the cost functions are heterogeneous. Then, each vote is informative in unbiased equilibria. Kitahara and Sekiguchi (2005b) showed that for any sequence of informative equilibria,

$$\Pr\Big(\sum_i y_i\big(q(x_i^n)\big) > n\Big) \to \Phi(\delta) \text{ as } n \to \infty,$$

where δ solves:

$$4\frac{\phi(\delta)}{\delta} = \frac{1}{\sum_k \alpha_k / \tilde{C}_k''(\frac{1}{2})}. \tag{7}$$

Therefore, similarly to the previous section, we obtain the following proposition:

Proposition 1 *Assume that for all k, $\tilde{C}_k'(1/2) = 0$. Then, for any sequence of unbiased equilibria $\{(\mathbf{x}^n, \mathbf{v}^n)\}$,*

(i) if $\tilde{C}''(1/2) = 0$ for some k, then

$$\Pr\Big(\sum_i y_i\big(q(x_i^n)\big) > n\Big) \to 1 \text{ as } n \to \infty;$$

(ii) otherwise,

$$\Pr\Big(\sum_i y_i\big(q(x_i^n)\big) > n\Big) \to \Phi(\delta) \text{ as } n \to \infty,$$

where δ solves Eq. (7).

This proposition asserts that CJT is valid if and only if there is *any* arbitrarily small fraction of voters who have cost functions with the second-order derivative at no information equal to zero.

5 Conclusions

In a symmetric two-alternatives election with costly information acquisition, general symmetric signal structures can be degenerated to the two-signals case if we assume that each vote is unbiased between the alternatives in equilibria.

For any unbiased voting equilibrium sequence, the probability of electing the desirable alternative converges to a unique value as the number of voters increases. If the second-order derivative of the information cost function is zero at no information, then this probability converges to one, that is, CJT is valid. Otherwise, it converges to $\Phi(\delta) < 1$, making the "rational ignorance" hypothesis valid.

We believe that our results are useful to various applications. For instance, Kitahara and Sekiguchi (2005a) considered the case in which signals are perturbed by normal noise, as in Example 2, and showed that if C' is concave, then the election leads to a poorer decision than in the case where choice is delegated to one of the voters. On the other hand, if C' is convex, then the election leads to a better decision if and only if the value of choosing the desirable alternative is sufficiently large.

Appendix

Proof of Lemma 1

Suppose that $(\mathbf{x}^*, \mathbf{v}^*)$ is an unbiased voting equilibrium. Then:

$$\begin{aligned}
\Pr\left(y_i = 1 \middle| z_L, (x_i^*, v_i^*)\right) &= E\left[v_i^*(s) \middle| z_L\right] \\
&= \int_S v_i^*(s) \mu_L^{x_i^*}(ds) \\
&= \int_S (1 - v_i^*(s)) \mu_R^{x_i^*}(ds) \\
&= E\left[1 - v_i^*(s) \middle| z_R\right] \\
&= \Pr\left(y_i = 1 \middle| z_R, (x_i^*, v_i^*)\right).
\end{aligned}$$

Let piv^i be the event that i's vote is pivotal. As $C(\bar{x}) = \infty$, $q(x_i^*) < 1$ for all i. Then, this event occurs with a positive probability in equilibria. As i's vote becomes pivotal if $\sum_{j \neq i} y_j^* = n$, we have:

$$\Pr\left(piv^i \middle| z_L, (\mathbf{x}_{-i}^*, \mathbf{v}_{-i}^*)\right) = \Pr\left(piv^i \middle| z_R, (\mathbf{x}_{-i}^*, \mathbf{v}_{-i}^*)\right) > 0. \quad (8)$$

Thus, in unbiased voting equilibria, the probability of i's vote being pivotal is the same as in both states. The probability distribution over states conditional on i receiving s and piv^i is computed by Bayes' rule. This is given by:

$$\begin{aligned}
\Pr\left(z \middle| piv^i, s, (\mathbf{x}^*, \mathbf{v}^*)\right) &= \frac{\Pr\left(z \middle| s, x_i^*\right) \Pr\left(piv^i \middle| z, (\mathbf{x}_{-i}^*, \mathbf{v}_{-i}^*)\right)}{\sum_{z' \in \{z_L, z_R\}} \Pr\left(z' \middle| s, x_i^*\right) \Pr\left(piv^i \middle| z', (\mathbf{x}_{-i}^*, \mathbf{v}_{-i}^*)\right)} \\
&= \Pr\left(z \middle| s, x_i^*\right).
\end{aligned}$$

The last equality is given by Eq. (8). Therefore, when a voter receives the signal s, the difference between the expected payoff from voting for L and for R is:

$$\Pr\left(z_L \middle| s, x_i^*\right) - \Pr\left(z_R \middle| s, x_i^*\right).$$

Consequently, i votes for L (R) if he or she is almost certain to receive $s \in S^L(x_i^*)$ ($S^R(x_i^*)$). Hence, we obtain Eq. (2).

Eq. (3) is immediately obtained from the definition of the unbiased strategy, and obviously we have $\Pr\left(y_i = 1 \middle| (x_i^*, v_i^*)\right) = q(x_i^*)$. □

Proof of Theorem 1

A voter i's expected payoff can be written as:

$$\begin{aligned}
&U_i(x_i, y_i | \mathbf{x}_{-i}, \mathbf{v}_{-i}) \\
&= \Pr\left(\sum_{j \neq i} y_j > n \middle| (\mathbf{x}_{-i}, \mathbf{v}_{-i})\right) + \Pr\left(\sum_{j \neq i} y_j = n, y_i = 1 \middle| (\mathbf{x}, \mathbf{v})\right) - C(x_i) \\
&= \Pr\left(\sum_{j \neq i} y_j = n \middle| (\mathbf{x}_{-i}, \mathbf{v}_{-i})\right) \Pr\left(y_i = 1 \middle| (x_i, v_i)\right) - C(x_i) + \text{Const.}
\end{aligned}$$

¿From Lemma 1, we obtain:

$$\begin{aligned}
&(x_i^*, v_i^*) \in \arg\max_{(x_i,v_i)} U_i(x_i, v_i|\mathbf{x}_{-i}, \mathbf{v}_{-i}), \ \forall i \\
\iff &\quad q(x_i^*) \in \arg\max_{q\in[1/2,1]} \left\{ q \Pr\Big(\sum_{j\neq i} y_j\big(q(x_j^*)\big) = n \Big) - \tilde{C}(q) \right\}, \ \forall i.
\end{aligned}$$

Obviously, the probability of winning the desirable alternative is equal to $\Pr\Big(\sum_i y_i\big(q(x_i^*)\big) > n \Big)$. □

Proof of Corollary 1

Suppose that $(\mathbf{x}^*, \mathbf{v}^*)$ is a strategy profile such that $x_i^* = \bar{x}_n$ for all i and $\mathbf{v}_i^*$ satisfies Eq. (2) (3). From Theorem 1, $(\mathbf{x}^*, \mathbf{v}^*)$ is an unbiased voting equilibrium if $\bar{x}_n$ solves Eq. (5). If $\tilde{C}'(1/2) = 0$, then such an $\bar{x}_n$ uniquely exists for all n. □

Proof of Theorem 2

Suppose that $\{\mathbf{x}^n, \mathbf{v}^n\}$ is a sequence of unbiased voting equilibria.

(i) Suppose that $\tilde{C}''(1/2) > 0$. In this case, it suffices to show that $q(x_i^n) = q(x_j^n)$ for all i, j if n is sufficiently large.

Suppose, on the contrary, that $q(x_i^n) \neq q(x_j^n)$. From the F.O.C.:

$$\begin{aligned}
\tilde{C}'(q(x_i^n)) &= \Pr\Big(\sum_{k\neq i} y_k\big(q(x_k^n)\big) = n \Big) \\
&= \Pr\Big(\sum_{k\neq i,j} y_k\big(q(x_k^n)\big) = n-1 \Big) \cdot q(x_j^n) \\
&\quad + \Pr\Big(\sum_{k\neq i,j} y_k\big(q(x_k^n)\big) = n \Big) \cdot \big(1 - q(x_j^n)\big).
\end{aligned}$$

Hence:

$$\begin{aligned}
&\frac{\tilde{C}'\big(q(x_i^n)\big) - \tilde{C}'\big(q(x_j^n)\big)}{q(x_i^n) - q(x_j^n)} \\
&\quad = \Pr\Big(\sum_{k\neq i,j} y_k\big(q(x_k^n)\big) = n \Big) - \Pr\Big(\sum_{k\neq i,j} y_k\big(q(x_k^n)\big) = n-1 \Big).
\end{aligned}$$

As $\tilde{C}''$ is continuous, the left-hand side is bounded away from 0. On the other hand, the right-hand side converges to 0, contradicting $q(x_i^n) \neq q(x_j^n)$.

(ii) Suppose that $\tilde{C}''(1/2) = 0$. If $q(x_i^n) = q$ for all i, then the probability that i's vote becomes pivotal is:

$$f_n(q) = \binom{2n}{n}\big(q(1-q)\big)^n.$$

It reaches a maximum at $q = 1/2$ and it is decreasing. We define $\overline{q}_n$ and $\underline{q}_n$ such that:

$$\begin{aligned}
\tilde{C}'(\overline{q}_n) &= f_n\left(\frac{1}{2}\right), \\
\tilde{C}'(\underline{q}_n) &= f_n\left(\overline{q}_n\right).
\end{aligned}$$

As $\tilde{C}'(q(x_i^n)) = \Pr\left(\sum_{j\neq i} y_j\big(q(x_j^n)\big) = n\right)$ from the F.O.C. and $\Pr\left(\sum_{j\neq i} y_j\big(q(x_j^n)\big) = n\right)$ is decreasing in $q(x_j^n)$, $q(x_i^n) \in [\underline{q}_n, \overline{q}_n]$ for all i and n.

Suppose that $\tilde{C}'(q) = \alpha(q - 1/2)^2$. In such a case, those bounds satisfy:

$$\left(\overline{q}_n - \frac{1}{2}\right) \simeq \left(\frac{1}{\alpha}\right)^{\frac{1}{2}} \left(\frac{1}{n\pi}\right)^{\frac{1}{4}}, \tag{9}$$

$$\left(\underline{q}_n - \frac{1}{2}\right) \simeq \left(\overline{q}_n - \frac{1}{2}\right) \exp\left(-\frac{1}{\alpha\sqrt{\pi}}\right). \tag{10}$$

for a sufficiently large n (from Starling's formula).

¿From Chebychev's inequality:

$$\sqrt{n}\left(q_n - \frac{1}{2}\right) \to \infty \Rightarrow \Pr\left(\sum_i y_i(q_n) > n\right) \to 1.$$

Obviously, $\sqrt{n}(q_n - 1/2) \to \infty$ if $\underline{q}_n$ satisfies Eq. (9) and (10) with equality. Then, even if all voters adopt the worst quality $\underline{q}_n$, $\Pr\left(\sum_j y_j\big(\underline{q}_n\big) > n\right) \to 1$.

Consider a general $\tilde{C}'$ that satisfies $\tilde{C}''(1/2) = 0$. If α is sufficiently large, $\tilde{C}'(1/2 + \epsilon) \leq \alpha\epsilon^2$ for a sufficiently small ϵ. Therefore, each voter's equilibrium quality is not less than $\underline{q}_n$ in the case of $\tilde{C}'(q) = \alpha(q - 1/2)^2$ for a sufficiently large n. As $\Pr(\sum_i y_i(q(x_i^n)) > n)$ is increasing in $q(x_i^n)$, CJT is valid for any sequence of equilibria if $\tilde{C}''(1/2) = 0$. □

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