Time Series Analysis

Nonstationary and Noninvertible Distribution Theory

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Chapter 12 Solutions to Problems

This chapter intends to present a complete set of solutions to problems given in the previous chapters. Most of the problems are concerned with corroborating the results described in the text. Thus this chapter will help making clear the details of discussions in the text.

Chapter 1.

1.1 It follows from (1.3) that

$$D_T = |(CC')^{-1} - \lambda I_T| = (2-\lambda)D_{T-1} - D_{T-2}$$

with $D_1 = 1-\lambda$ and $D_2 = (2-\lambda)(1-\lambda) - 1$. Then we have

$$D_T = \frac{1-x_1}{x_2-x_1}x_1^T + \frac{x_2-1}{x_2-x_1}x_2^T = \frac{\cos(T+\frac{1}{2})\omega}{\cos\frac{\omega}{2}} \ ,$$

where x_1 and x_2 are the solutions to $x^2 - (2-\lambda)x + 1 = 0$. We can put $x_1 = e^{i\omega}$ and $x_2 = e^{-i\omega}$ so that $\cos\omega = 1 - \lambda/2$ and $\sin\omega = \sqrt{4\lambda - \lambda^2}/2$. The solutions to $D_T = 0$ are those to $\cos\left(T + \dfrac{1}{2}\right)\omega = 0$, which yields $\omega = \left(t - \dfrac{1}{2}\right)\pi \Big/ \left(T + \dfrac{1}{2}\right)$ and thus $\lambda = 2 - 2\cos\omega = 4\sin^2\dfrac{\omega}{2}$.

1.2 We have $\phi_T(-i\theta) = (D_T(\theta))^{-\frac{1}{2}}$, where

$$D_T(\theta) = \left| I_T - \frac{2\theta}{\left(T+\frac{1}{2}\right)^2}CC'\right| = \left|(CC')^{-1} - \frac{2\theta}{\left(T+\frac{1}{2}\right)^2}I_T\right| .$$

Then it holds that

$$D_T(\theta) = \cos T\omega + \frac{(a-2)\sin T\omega}{2\sin\omega} \ ,$$

where $a = 2 - 2\theta \Big/ \left(T + \dfrac{1}{2}\right)^2$, $\cos\omega = a/2$ and $\sin\omega = \sqrt{4-a^2}/2$. We now obtain

$$T\omega = T\tan^{-1}\frac{\sqrt{4-a^2}}{a} = \sqrt{2\theta}\left(1 - \frac{1}{2T} + \frac{\theta+3}{12T^2}\right) + O\left(\frac{1}{T^3}\right) \ ,$$

$$\cos T\omega = \cos\sqrt{2\theta} + \frac{\sqrt{2\theta}}{2T}\sin\sqrt{2\theta} - \frac{1}{4T^2}\left(\theta\cos\sqrt{2\theta} + \sqrt{2\theta}\left(1+\frac{\theta}{3}\right)\sin\sqrt{2\theta}\right) + O\left(\frac{1}{T^3}\right) \ ,$$

$$\sin T\omega = \sin\sqrt{2\theta} - \frac{\sqrt{2\theta}}{2T}\cos\sqrt{2\theta} + O\left(\frac{1}{T^2}\right) \ ,$$

$$\frac{a-2}{2\sin\omega} = \frac{a-2}{\sqrt{4-a^2}} = -\frac{\sqrt{2\theta}}{2T}(1-\frac{1}{2T}) + O\left(\frac{1}{T^3}\right) .$$

Substitution of these into $D_T(\theta)$ yields

$$D_T(\theta) = \cos\sqrt{2\theta}\left[1+\frac{\theta}{4T^2}\left(1-\frac{\sqrt{2\theta}}{3}\tan\sqrt{2\theta}\right)\right] + O\left(\frac{1}{T^3}\right) ,$$

and thus we arrive at the desired result.

1.3 Defining $\sum(\rho) = I_T + \rho CC'$ and noting that $z = \sum^{-\frac{1}{2}}(\rho_0)y \sim \mathrm{N}(0, I_T)$ under H_0, we have, from (1.5),

$$\begin{aligned} \left.\frac{dL(\rho)}{d\rho}\right|_{\rho=\rho_0} &= -\frac{1}{2}\operatorname{tr}\left(\textstyle\sum^{-1}(\rho_0)CC'\right) + \frac{1}{2}z'\textstyle\sum^{-\frac{1}{2}}(\rho_0)CC'\textstyle\sum^{-\frac{1}{2}}(\rho_0)z \\ &= \frac{1}{2}\sum_{t=1}^{T}\frac{\lambda_t}{1+\rho_0\lambda_t}(\xi_t^2-1) , \end{aligned}$$

where $\xi = (\xi_1, \cdots, \xi_T)' \sim \mathrm{N}(0, I_T)$. The LM test for the present problem rejects H_0 for large values of the above statistic. By the Lyapunov central limit theorem it can be shown that

$$\frac{1}{\sqrt{T}}\left.\frac{dL(\rho)}{d\rho}\right|_{\rho=\rho_0} \longrightarrow \mathrm{N}(0, \sigma^2) ,$$

where

$$\begin{aligned} \sigma^2 &= \frac{1}{2}\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\frac{\lambda_t^2}{(1+\rho_0\lambda_t)^2} = \frac{1}{2\pi}\int_0^{\frac{\pi}{2}}\frac{dx}{(\rho_0+4\sin^2 x)^2} \\ &= \frac{\rho_0+2}{4(\rho_0(\rho_0+4))^{3/2}} . \end{aligned}$$

1.4 Putting $x = \left(t-\frac{1}{2}\right)\pi/(2T+1)$ we have

$$\frac{4\lambda_t}{(2T+1)^2} - \frac{1}{\left(t-\frac{1}{2}\right)^2\pi^2} = \frac{1}{(2T+1)^2}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right) ,$$

which is positive and increasing for $0 < x < \pi/2$. Thus, for any $\delta > 0$,

$$P\left(\sum_{t=1}^{T}\left(\frac{4\lambda_t}{(2T+1)^2} - \frac{1}{\left(t-\frac{1}{2}\right)^2\pi^2}\right)\xi_t^2 \geq \delta\right)$$

$$\le \frac{T}{\delta}\left(\frac{4\lambda_T}{(2T+1)^2} - \frac{1}{\left(T-\frac{1}{2}\right)^2\pi^2}\right)$$

$$= \frac{T}{\delta}\left(\frac{1}{(2T+1)^2}\left(\sin\frac{T-\frac{1}{2}}{2T+1}\pi\right)^{-2} - \frac{1}{\left(T-\frac{1}{2}\right)^2\pi^2}\right) \longrightarrow 0\,,$$

which leads us to (1.8).

1.5 Put $F(x) = P(W \le x)$ and $f(x) = F'(x)$. Then

$$\begin{aligned}\int_0^\infty e^{-\theta x} f(x)dx &= (\cos\sqrt{-2\theta})^{-\frac{1}{2}} = (\cosh\sqrt{2\theta})^{-\frac{1}{2}} \\ &= \sqrt{2}e^{-\sqrt{\theta/2}}\left(1+e^{-2\sqrt{2\theta}}\right)^{-\frac{1}{2}} \\ &= \sqrt{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n} e^{-(2n+\frac{1}{2})\sqrt{2\theta}}\end{aligned}$$

so that, taking the inverse Laplace transform,

$$\begin{aligned}f(x) &= \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\frac{\sqrt{2}}{2\sqrt{\pi x^3}} b_n e^{-b_n^2/(4x)}\,, \quad \left(b_n = \sqrt{2}\left(2n+\frac{1}{2}\right)\right)\,, \\ F(x) &= \int_0^x f(a)da \\ &= \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\frac{2}{\sqrt{\pi}}\int_{b_n/\sqrt{2x}}^{\infty} e^{-a^2/2}da \\ &= 2\sqrt{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\Phi\left(-\frac{2n+\frac{1}{2}}{\sqrt{x}}\right)\,.\end{aligned}$$

2.1 By the definition of Ω we have

$$D_T = |\Omega - \lambda I_T| = (2-\lambda)D_{T-1} - D_{T-2}$$

with $D_1 = 2-\lambda$ and $D_2 = (2-\lambda)^2 - 1$. Proceeding in the same way as in the solution to Problem 1.1, we obtain $D_T = \sin(T+1)\omega/\sin\omega$ with $\cos\omega = 1-\lambda/2$ and

$\sin\omega = \sqrt{4\lambda - \lambda^2}/2$. Thus the solutions to $D_T = 0$ are those to $\sin(T+1)\omega = 0$, which yields $\omega = t\pi/(T+1)$ and $\lambda = 2 - 2\cos\omega = 4\sin^2\dfrac{\omega}{2}$.

2.2 Put $\phi_T(-i\theta) = (\tilde{D}_T(\theta))^{-\frac{1}{2}}$. Then

$$\begin{aligned}\tilde{D}_T(\theta) &= \left| I_T - \frac{2\theta}{(T+1)^2}\Omega^{-1} \right| = |\Omega|^{-1}\left| \Omega - \frac{2\theta}{(T+1)^2} I_T \right| \\ &= \frac{1}{T+1}\left(\cos T\omega + \frac{a\sin T\omega}{2\sin\omega} \right),\end{aligned}$$

where $a = 2 - 2\theta/(T+1)^2$, $\cos\omega = a/2$ and $\sin\omega = \sqrt{4-a^2}/2$. We now obtain

$$T\omega = T\tan^{-1}\frac{\sqrt{4-a^2}}{a} = \sqrt{2\theta}\left(1 - \frac{1}{T} + \frac{\theta+12}{12T^2}\right) + O\left(\frac{1}{T^3}\right),$$

$$\frac{\cos T\omega}{T+1} = \frac{1}{T}\cos\sqrt{2\theta} - \frac{1}{T^2}(\cos\sqrt{2\theta} - \sqrt{2\theta}\sin\sqrt{2\theta}) + O\left(\frac{1}{T^3}\right),$$

$$\sin T\omega = \sin\sqrt{2\theta} - \frac{\sqrt{2\theta}}{T}\cos\sqrt{2\theta} + \frac{1}{T^2}\left(\frac{\theta+12}{12}\sqrt{2\theta}\cos\sqrt{2\theta} - \theta\sin\sqrt{2\theta}\right) + O\left(\frac{1}{T^3}\right),$$

$$\frac{a}{2(T+1)\sin\omega} = \frac{a}{(T+1)\sqrt{4-a^2}} = \frac{1}{\sqrt{2\theta}} - \frac{3\sqrt{2\theta}}{8T^2} + O\left(\frac{1}{T^3}\right).$$

Substitution of these into $\tilde{D}_T(\theta)$ yields

$$\tilde{D}_T(\theta) = \frac{\sin\sqrt{2\theta}}{\sqrt{2\theta}}\left[1 + \frac{\theta}{4T^2}\left(1 + \frac{\sqrt{2\theta}}{3}\cot\sqrt{2\theta}\right)\right] + O\left(\frac{1}{T^3}\right)$$

and thus we arrive at the conclusion.

2.3 Putting $x = t\pi/(2(T+1))$ we can proceed completely in the same way as in the solution to Problem 1.4 and establish (1.16).

2.4 We have $\varepsilon'\Omega^{-2}\varepsilon = \sum_{t=1}^{T}\delta_t^2\xi_t$, where $\xi_t \sim \mathrm{NID}(0,1)$ and δ_t is defined in (1.14). It can be proved, as in the solution to Problem 1.4, that

$$\operatorname*{plim}_{T\to\infty}\left(\frac{1}{T^4}\sum_{t=1}^{T}\delta_t^2\xi_t^2 - \sum_{t=1}^{T}\frac{\xi_t^2}{t^4\pi^4}\right) = 0$$

so that

$$\mathcal{L}\left(\frac{1}{T^4}\varepsilon'\Omega^{-2}\varepsilon\right) = \mathcal{L}\left(\frac{1}{T^4}\sum_{t=1}^{T}\delta_t^2\xi_t^2\right) \longrightarrow \mathcal{L}\left(\sum_{n=1}^{\infty}\frac{\xi_n^2}{n^4\pi^4}\right)$$

and thus the c.f. $\phi(\theta)$ of this last random variable is

$$\begin{aligned}
\phi(\theta) &= \prod_{n=1}^{\infty}\left(1-\frac{2i\theta}{n^4\pi^4}\right)^{-\frac{1}{2}} = \prod_{n=1}^{\infty}\left(1-\frac{\sqrt{2i\theta}}{n^2\pi^2}\right)^{-\frac{1}{2}}\prod_{n=1}^{\infty}\left(1+\frac{\sqrt{2i\theta}}{n^2\pi^2}\right)^{-\frac{1}{2}} \\
&= \left(\frac{\sin(2i\theta)^{\frac{1}{4}}}{(2i\theta)^{\frac{1}{4}}}\right)^{-\frac{1}{2}}\left(\frac{\sinh(2i\theta)^{\frac{1}{4}}}{(2i\theta)^{\frac{1}{4}}}\right)^{-\frac{1}{2}}.
\end{aligned}$$

2.5 The log-likelihood $L(\alpha,\sigma^2)$ for y is

$$L(\alpha,\sigma^2) = -\frac{T}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}y'\Phi^{-1}(\alpha)y\,,$$

where $\Phi(\alpha)$ is the same as $\Omega(\alpha)$ except the (1,1) element, which is unity. Then it is easy to obtain

$$\begin{aligned}
\left.\frac{\partial L(\alpha,\sigma^2)}{\partial\alpha}\right|_{\alpha=1,\sigma^2=\hat\sigma^2} &= \frac{1}{2\hat\sigma^2}y'C'C\left[C^{-1}(C^{-1})'-e_1e_1'\right]C'Cy \\
&= \frac{1}{2\hat\sigma^2}y'[C'C-C'ee'C]y \\
&= \frac{T}{2}\left[1-\frac{y'C'ee'Cy}{y'C'Cy}\right],
\end{aligned}$$

where $\hat\sigma^2 = y'C'Cy/T$ with C defined in (1.3), $e_1 = (1,0,\cdots,0)'$ and $e=(1,\cdots,1)'$. The LM test rejects H_0 for small values of $\partial L(\alpha,\sigma^2)/\partial\alpha|_{\alpha=1,\,\sigma^2=\hat\sigma^2}$, which is equivalent to rejecting H_0 for large values of

$$S_T = \frac{y'C'ee'Cy}{y'C'Cy}\,.$$

Noting that $y \sim \mathrm{N}(0,\sigma^2C^{-1}(C^{-1})')$ under H_0, we have that S_T is asymptotically distributed as $\chi^2(1)$ under H_0.

2.6 The log-likelihood $L(\alpha,\sigma^2)$ for y is

$$L(\alpha,\sigma^2) = -\frac{T}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}y'(C')^{-1}\Phi^{-1}(\alpha)C^{-1}y\,,$$

where $\Phi(\alpha)$ is defined in the solution to Problem 2.5. Then we obtain

$$\begin{aligned}
\left.\frac{\partial L(\alpha,\sigma^2)}{\partial\alpha}\right|_{\alpha=1,\sigma^2=\hat\sigma^2} &= \frac{1}{2\hat\sigma^2}y'(C')^{-1}C'C\left[C^{-1}(C^{-1})'-e_1e_1'\right]C'CC^{-1}y \\
&= \frac{1}{2\hat\sigma^2}y'[I_T-ee']y \\
&= \frac{T}{2}\left[1-\frac{y'ee'y}{y'y}\right],
\end{aligned}$$

where $\hat\sigma^2 = y'y/T$. The LM test rejects H_0 for small values of the above quantity, which is equivalent to rejecting H_0 for large values of $y'ee'y/y'y = \left(\sum_{t=1}^{T} y_t\right)^2 \Big/ \sum_{t=1}^{T} y_t^2$.

2.7 The log-likelihood $L(\alpha,\mu,\sigma^2)$ for y is

$$L(\alpha,\mu,\sigma^2) = -\frac{T}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}(y-\mu e)'(C')^{-1}\Phi^{-1}(\alpha)C^{-1}(y-\mu e)\,.$$

The LM test considered here rejects H_0 for large values of

$$\begin{aligned}
\frac{\partial^2 L(1,\hat\mu,\hat\sigma^2)}{\partial\alpha^2} &= \frac{-1}{2\hat\sigma^2}y'M(C')^{-1}\frac{\partial^2\Phi^{-1}(1)}{\partial\alpha^2}C^{-1}My \\
&= \frac{1}{\hat\sigma^2}y'M(C')^{-1}\left[C'(CC'-ee')C - C'(I_T-ee')^2C\right]C^{-1}My \\
&= T\left[\frac{y'MCC'My}{y'My}-1\right],
\end{aligned}$$

where $\hat\mu = \bar y$, $\hat\sigma^2 = y'My/T$ and $M = I_T - ee'/T$. Thus the LM test is equivalent to rejecting H_0 for large values of SL_T in (1.20).

2.8 Consider first

$$\begin{aligned}
y'My &= (C^{-1}y)'C'MC(C^{-1}y) \\
&= \begin{pmatrix} y_1 \\ \Delta y\end{pmatrix}' C'MC \begin{pmatrix} y_1 \\ \Delta y\end{pmatrix},
\end{aligned}$$

where it can be checked that

$$C'MC = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & \Omega_*^{-1} & \\ 0 & & & & \end{pmatrix}$$

so that $y'My = (\Delta y)'\Omega_*^{-1}(\Delta y)$. Similarly, we have

$$\begin{aligned} y'MCC'My &= (C^{-1}y)'C'MCC'MC(C^{-1}y) \\ &= (\Delta y)'\Omega_*^{-2}(\Delta y)\,, \end{aligned}$$

which establishes (1.21).

3.1 The first component of $C'M\varepsilon$ is 0 so that

$$C'M\varepsilon = \begin{pmatrix} 1 & \cdots & 1 \\ 0 & C'_* & \end{pmatrix} (\varepsilon - \bar{\varepsilon}e) = \begin{pmatrix} 0 \\ C'_*(\varepsilon_* - \bar{\varepsilon}e_*) \end{pmatrix},$$

where C'_*, ε_* and e_* are the last $(T-1)\times(T-1)$, $(T-1)\times 1$ and $(T-1)\times 1$ submatrices of C', ε and e, respectively. Noting that

$$\mathcal{L}(\varepsilon_* - \bar{\varepsilon}e_*) = \mathcal{L}\left(\left(I_{T-1} - \frac{1}{T}e_*e'_*\right)^{1/2}\xi_*\right),$$

where $\xi_* \sim \mathrm{N}(0, I_{T-1})$, we have

$$\begin{aligned} \mathcal{L}(\varepsilon'MCC'M\varepsilon) &= \mathcal{L}((\varepsilon_* - \bar{\varepsilon}e_*)'C_*C'_*(\varepsilon_* - \bar{\varepsilon}e_*)) \\ &= \mathcal{L}\left(\xi'_*C'_*\left(I_{T-1} - \frac{1}{T}e_*e'_*\right)C_*\xi_*\right) \\ &= \mathcal{L}(\xi'_*\Omega_*^{-1}\xi_*) \end{aligned}$$

with Ω_* being the first $(T-1)\times(T-1)$ submatrix of Ω. Therefore (1.23) has been established.

3.2 Put $\phi_T(\theta) = a(\theta) + ib(\theta)$, where $a(\theta)$ and $b(\theta)$ are real and $a(\theta) = a(-\theta)$, $b(\theta) = -b(-\theta)$ because of the property of $\phi_T(\theta)$. Then we have

$$\begin{aligned} f_T(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}(a(\theta)\cos\theta x + b(\theta)\sin\theta x)d\theta \\ &= \frac{1}{\pi}\int_0^{\infty}(a(\theta)\cos\theta x + b(\theta)\sin\theta x)d\theta = \frac{1}{\pi}\int_0^{\infty}\mathrm{Re}\left(e^{-i\theta x}\phi_T(\theta)\right)d\theta\,. \end{aligned}$$

3.3 Consider first (1.7) and put

$$\begin{aligned}
\psi_{1T}(\theta) &= \log \phi_T(-i\theta) \\
&\sim -\frac{1}{2}\log\cos\sqrt{2\theta} + \log\left[1 - \frac{\theta}{8T^2}\left(1 - \frac{\sqrt{2\theta}}{3}\tan\sqrt{2\theta}\right)\right] \\
&\sim -\frac{1}{2}\log(1-g(\theta)) - \frac{\theta}{8T^2}\left(1 - \frac{\sqrt{2\theta}}{3}\tan\sqrt{2\theta}\right) \\
&\sim \frac{1}{2}\left(g(\theta) + \frac{g^2(\theta)}{2} + \frac{g^3(\theta)}{3} + \frac{g^4(\theta)}{4} + \cdots\right) - \frac{\theta}{8T^2} \\
&\qquad + \frac{\theta}{24T^2}\left(2\theta + \frac{4\theta^2}{3} + \frac{16\theta^3}{15} + \frac{272\theta^4}{315} + \cdots\right),
\end{aligned}$$

where

$$g(\theta) = \frac{2\theta}{2!} - \frac{4\theta^2}{4!} + \frac{8\theta^3}{6!} - \frac{16\theta^4}{8!} + \cdots.$$

Consider next (1.15) and put

$$\begin{aligned}
\psi_{2T}(\theta) &= \log \phi_T(-i\theta) \\
&\sim -\frac{1}{2}\log\frac{\sin\sqrt{2\theta}}{\sqrt{2\theta}} + \log\left[1 - \frac{\theta}{8T^2}\left(1 + \frac{\sqrt{2\theta}}{3}\cot\sqrt{2\theta}\right)\right] \\
&\sim -\frac{1}{2}\log(1-h(\theta)) - \frac{\theta}{8T^2}\left(1 + \frac{\sqrt{2\theta}}{3}\cot\sqrt{2\theta}\right) \\
&\sim \frac{1}{2}\left(h(\theta) + \frac{h^2(\theta)}{2} + \frac{h^3(\theta)}{3} + \frac{h^4(\theta)}{4} + \cdots\right) - \frac{\theta}{8T^2} \\
&\qquad - \frac{\theta}{24T^2}\left(1 - \frac{2\theta}{3} - \frac{4\theta^2}{45} - \frac{16\theta^3}{945} - \cdots\right),
\end{aligned}$$

where

$$h(\theta) = \frac{2\theta}{3!} - \frac{4\theta^2}{5!} + \frac{8\theta^3}{7!} - \frac{16\theta^4}{9!} + \cdots.$$

Evaluating $\kappa_{iT}^{(j)} = d^j\psi_{iT}(0)/d\theta^j$ $(i = 1, 2;\ j = 1, 2, 3, 4)$ we arrive at (1.28).

3.4 As for $\kappa_1^{(j)}$ consider

$$\psi_1(\theta) = -\frac{1}{2}\log\cos\sqrt{2\theta} = -\frac{1}{2}\sum_{n=1}^{\infty}\log\left(1 - \frac{2\theta}{\left(n-\frac{1}{2}\right)^2\pi^2}\right)$$

$$= \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \frac{2^{j-1}\theta^j}{j\left(\left(n-\frac{1}{2}\right)\pi\right)^{2j}} = \sum_{j=1}^{\infty} \frac{\theta^j}{j!}(j-1)!\, 2^{3j-1} \sum_{n=1}^{\infty} \frac{1}{((2n-1)\pi)^{2j}} .$$

By the definition of the Bernoulli numbers we have

$$B_j = \frac{(2j)!}{2^{2j-1}} \sum_{n=1}^{\infty} \frac{1}{(n\pi)^{2j}} ,$$

so that

$$\sum_{n=1}^{\infty} \frac{1}{((2n-1)\pi)^{2j}} = \frac{1}{2(2j)!}(2^{2j}-1)B_j .$$

Therefore we obtain

$$\psi_1(\theta) = \sum_{j=1}^{\infty} \frac{\theta^j}{j!} \frac{(j-1)!\, 2^{3j-2}(2^{2j}-1)}{(2j)!} B_j ,$$

which yields $\kappa_1^{(j)}$ in (1.29). The expression for $\kappa_2^{(j)}$ can be obtained similarly.

3.5 It is clear that

$$V_T = \frac{1}{T^2}\, y' \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & \delta \end{pmatrix} y = \frac{1}{T^2} \varepsilon' C' \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & \delta \end{pmatrix} C\varepsilon ,$$

while

$$\begin{aligned} U_T &= \frac{1}{T}\sum_{t=2}^{T} y_{t-1}(y_t - y_{t-1}) - \frac{\delta}{T} y_T^2 \\ &= -\frac{1}{2T}\left(\sum_{t=2}^{T}(y_t - y_{t-1})^2 - \sum_{t=2}^{T} y_t^2 + \sum_{t=2}^{T} y_{t-1}^2\right) - \frac{\delta}{T} y_T^2 \\ &= -\frac{1}{2T}\left(\sum_{t=2}^{T} \varepsilon_t^2 - y_T^2 + y_1^2\right) - \frac{\delta}{T} y_T^2 \\ &= \frac{1-2\delta}{2T} y_T^2 - \frac{1}{2T}\sum_{t=1}^{T} \varepsilon_t^2 , \end{aligned}$$

which leads us to the conclusion.

3.6 For any α, β and γ let us consider

$$\begin{aligned} n_T(\theta) &= \left| I_T - 2\theta\left\{\alpha C' \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & \delta \end{pmatrix} C + \beta\, ee' + \gamma I_T \right\}\right|^{-\frac{1}{2}} \\ &= D_T^{-\frac{1}{2}}, \end{aligned}$$

where

$$D_T = \left| a(CC')^{-1} + b \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & \delta \end{pmatrix} + c e_T e_T' \right|,$$

$$a = 1 - 2\gamma\theta\,, \quad b = -2\alpha\theta\,, \quad c = -2\beta\theta\,.$$

Since $D_T = (2a+b)D_{T-1} - a^2 D_{T-2}$ with $D_1 = a + b\delta + c$ and $D_2 = (2a+b)(a+b\delta+c) - a^2$, it can be shown that

$$D_T = \frac{1}{x_2 - x_1}\left[(x_2 - (a + b\delta + c))x_1^T + (a + b\delta + c - x_1)x_2^T\right],$$

where x_1 and x_2 are solutions to $x^2 - (2a+b)x + a^2 = 0$. Putting $\alpha = x/T^2$, $\beta = (2\delta - 1)/(2T)$ and $\gamma = 1/(2T)$, and expressing x_1 and x_2 in polar form, we arrive at (1.33).

3.7 In the solution to Problem 3.6 we have $x_1 = re^{i\omega}$ and $x_2 = re^{-i\omega}$, where

$$r = 1 - \frac{\theta}{T}\,, \quad r\cos\omega = 1 - \frac{\theta}{T} - \frac{\theta x}{T^2}\,, \quad r\sin\omega = \frac{1}{T}\sqrt{2\theta x\left(1 - \frac{\theta}{T} - \frac{\theta x}{2T^2}\right)}\,.$$

Therefore we have

$$\begin{aligned} r^T &= \left(1 - \frac{\theta}{T}\right)^T = \exp\left\{T\log\left(1 - \frac{\theta}{T}\right)\right\} \\ &= \left(1 - \frac{\theta^2}{2T}\right)e^{-\theta} + O\left(\frac{1}{T^2}\right), \end{aligned}$$

$$
\begin{aligned}
T\omega &= T\tan^{-1}\left[\frac{1}{T}\left(1-\frac{\theta}{T}-\frac{\theta x}{T^2}\right)^{-1}\left(2\theta x\left(1-\frac{\theta}{T}-\frac{\theta x}{2T^2}\right)\right)^{\frac{1}{2}}\right] \\
&= \sqrt{2\theta x}+\frac{\theta\sqrt{2\theta x}}{2T}+O\left(\frac{1}{T^2}\right),
\end{aligned}
$$

$$
\begin{aligned}
\cos T\omega &= \cos\sqrt{2\theta x}-\frac{\theta\sqrt{2\theta x}}{2T}\sin\sqrt{2\theta x}+O\left(\frac{1}{T^2}\right), \\
\sin T\omega &= \sin\sqrt{2\theta x}+\frac{\theta\sqrt{2\theta x}}{2T}\cos\sqrt{2\theta x}+O\left(\frac{1}{T^2}\right),
\end{aligned}
$$

$$
\begin{aligned}
\frac{r\cos\omega-d}{r\sin\omega} &= T\left(1-\frac{\theta}{T}-\frac{\theta x}{T^2}-\left(1-\frac{2\delta\theta}{T}-\frac{2\delta\theta x}{T^2}\right)\right)\left(2\theta x\left(1-\frac{\theta}{T}-\frac{\theta x}{2T^2}\right)\right)^{-\frac{1}{2}} \\
&= -\frac{\theta(1-2\delta)}{\sqrt{2\theta x}}-\frac{\theta(\theta(1-2\delta)+2x-4\delta x)}{2T\sqrt{2\theta x}}+O\left(\frac{1}{T^2}\right).
\end{aligned}
$$

Substituting these into $m_T(\theta)$ in (1.33) we arrive at (1.34).

3.8 Let the random variable on the right side of (1.42) be $Y_T = A_T/B_T$. Consider then $P(Y_T \le x) = P(xB_T - A_T \ge 0)$, where

$$
xB_T - A_T = \frac{x}{T^2}\sum_{t=2}^{T} y_{t-1}^2 - \frac{1}{T}\sum_{t=2}^{T} y_{t-1}\varepsilon_t - \frac{1}{2}.
$$

Comparing with (1.32) and (1.35) it is easily seen that the limiting c.f. $\phi(\theta;x)$ of this last random variable is given by

$$
\phi(\theta;x) = \left[\cos\sqrt{2i\theta x}+i\theta\,\frac{\sin\sqrt{2i\theta x}}{\sqrt{2i\theta x}}\right]^{-\frac{1}{2}}
$$

so that

$$
\begin{aligned}
\psi(\theta_1,-\theta_2) &= \phi(i\theta_1;\theta_2/\theta_1) \\
&= \left[\cosh\sqrt{2\theta_2}-\theta_1\frac{\sinh\sqrt{2\theta_2}}{\sqrt{2\theta_2}}\right]^{-\frac{1}{2}}.
\end{aligned}
$$

Thus $E(Y)$, where $\mathcal{L}(Y_T) \longrightarrow \mathcal{L}(Y)$, is given by

$$
\begin{aligned}
E(Y) &= \int_0^\infty \left.\frac{\partial\psi(\theta_1,-\theta_2)}{\partial\theta_1}\right|_{\theta_1=0} d\theta_2 \\
&= \frac{1}{2}\int_0^\infty \frac{\sinh\sqrt{2\theta}}{\sqrt{2\theta}}(\cosh\sqrt{2\theta})^{-\frac{3}{2}}\,d\theta \\
&= 1,
\end{aligned}
$$

where the last equality results from the second by putting $(\cosh\sqrt{2\theta})^{-\frac{1}{2}} = u$.

3.9 From (1.40) we have $\mu_{1-\delta}(1) = \mu_\delta(1) - 2(1-2\delta)$ so that (1.43) is equivalent to

(S1) $$\mu_{1-\delta}(2) - \mu_\delta(2) = 4(1-2\delta)(1-2\delta-\mu_\delta(1))\,.$$

Putting

$$\begin{aligned}
\psi_\delta(\theta_1, -\theta_2) &= \phi_\delta(i\theta_1; \theta_2/\theta_1) \\
&= e^{-\frac{\theta_1}{2}}\left[\cosh\sqrt{2\theta_2} - \theta_1(1-2\delta)\frac{\sinh\sqrt{2\theta_2}}{\sqrt{2\theta_2}}\right]^{-\frac{1}{2}},
\end{aligned}$$

it can be shown from (1.39) that the left side of (S1) is equal to

$$\begin{aligned}
&\int_0^\infty \theta_2 \frac{\partial^2}{\partial\theta_1^2}\left(\psi_{1-\delta}(\theta_1, -\theta_2) - \psi_\delta(\theta_1, -\theta_2)\right)\Big|_{\theta_1=0} d\theta_2 \\
&\qquad = (1-2\delta)\int_0^\infty \theta\frac{\sinh\sqrt{2\theta}}{\sqrt{2\theta}}(\cosh\sqrt{2\theta})^{-\frac{3}{2}} d\theta\,,
\end{aligned}$$

while, from the solution to Problem 3.8,

$$\begin{aligned}
\mu_\delta(1) &= -\frac{1}{2}\int_0^\infty \frac{1}{\sqrt{\cosh\sqrt{2\theta_1}}}\left(1 - \frac{1-2\delta}{\cosh\sqrt{2\theta_1}}\frac{\sinh\sqrt{2\theta_1}}{\sqrt{2\theta_1}}\right) d\theta_1 \\
&= -\frac{1}{2}\int_0^\infty \frac{d\theta}{\sqrt{\cosh\sqrt{2\theta}}} + (1-2\delta) \\
&= -\frac{1}{4}\int_0^\infty \theta\,\frac{\sinh\sqrt{2\theta}}{\sqrt{2\theta}}(\cosh\sqrt{2\theta})^{-\frac{3}{2}} d\theta + (1-2\delta).
\end{aligned}$$

Substituting this last expression into the right side of (S1) we arrive at the conclusion.

4.1 We have

$$\begin{aligned}
\phi_T(\theta) &= \left|I_{2T} - \frac{i\theta}{T^2}\begin{pmatrix} 0 & C'C \\ C'C & 0 \end{pmatrix}\right|^{-\frac{1}{2}} \\
&= \left|\begin{pmatrix} I_T & -\dfrac{i\theta}{T^2}C'C \\ -\dfrac{i\theta}{T^2}C'C & I_T \end{pmatrix}\right|^{-\frac{1}{2}} \\
&= \left|I_T + \frac{\theta^2}{T^4}(C'C)^2\right|^{-\frac{1}{2}},
\end{aligned}$$

which yields the result.

4.2 We rewrite V_T as

$$\begin{aligned} V_T &= \frac{1}{T}\sum_{t=1}^{T} y_{1t}(y_{1t} - y_{1,t-1}) \\ &= \frac{1}{2T}\left(\sum_{t=1}^{T}(y_{1t} - y_{1,t-1})^2 + \sum_{t=1}^{T} y_{1t}^2 - \sum_{t=1}^{T} y_{1,t-1}^2\right) \\ &= \frac{1}{2T}\sum_{t=1}^{T}\varepsilon_{1t}^2 + \frac{1}{2}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{1t}\right)^2 . \end{aligned}$$

Here the first term converges in probability to 1/2, while the second converges in distribution to $\xi^2/2$, which establishes the result.

4.3 As for $\phi_{1T}(\theta)$ we can proceed in the same way as in the solution to Problem 4.1. Put $e = (1, \cdots, 1)' : T \times 1$ and we have

$$\begin{aligned} \phi_{1T}(\theta) &= \left|\left(I_T + \frac{i\theta}{2T}(C' + C)\right)\left(I_T - \frac{i\theta}{2T}(C' + C)\right)\right|^{-\frac{1}{2}} \\ &= \left|\left(\left(1 + \frac{i\theta}{2T}\right)I_T + \frac{i\theta}{2T}ee'\right)\left(\left(1 - \frac{i\theta}{2T}\right)I_T - \frac{i\theta}{2T}ee'\right)\right|^{-\frac{1}{2}} \\ &= \left|\left(1 + \frac{i\theta}{2T}\right)^{T-1}\left(1 + \frac{i\theta}{2} + \frac{i\theta}{2T}\right)\left(1 - \frac{i\theta}{2T}\right)^{T-1}\left(1 - \frac{i\theta}{2} - \frac{i\theta}{2T}\right)\right|^{-\frac{1}{2}} , \end{aligned}$$

which yields (1.54). As for $\phi_{2T}(\theta)$ put $a = \theta/(2T)$ and consider

$$\phi_{2T}(\theta) = (D_T(a)D_T(-a))^{-\frac{1}{2}} ,$$

where

$$\begin{aligned} D_T(a) &= |I_T + aC' - aC| \\ &= |(C')^{-1} + aI_T - a(C')^{-1}C| \\ &= (1 + a)D_{T-1}(a) - a(1 - a)^{T-1} . \end{aligned}$$

Noting that $D_1(a) = 1$ we can solve the above difference equation as

$$D_T(a) = \frac{1}{2}\left((1 + a)^T + (1 - a)^T\right) ,$$

which leads us to (1.55).

4.4 It is easy to see that

$$\begin{aligned}\psi_1(\theta) &= \log \phi_1(-i\theta) = -\frac{1}{2}\log\left(1-\frac{\theta^2}{4}\right) \\ &= \sum_{j=1}^{\infty}\frac{\theta^{2j}}{(2j)!}\,\frac{(2j)!}{j\,2^{2j+1}}\,,\end{aligned}$$

$$\begin{aligned}\psi_2(\theta) &= \log \phi_2(-i\theta) = -\sum_{n=1}^{\infty}\log\left(1-\frac{\theta^2}{((2n-1)\pi)^2}\right) \\ &= \sum_{j=1}^{\infty}\frac{\theta^{2j}}{(2j)!}\,\frac{(2j)!}{j}\sum_{n=1}^{\infty}\frac{1}{((2n-1)\pi)^{2j}} \\ &= \sum_{j=1}^{\infty}\frac{\theta^{2j}}{(2j)!}\,\frac{2^{2j}-1}{2j}\,B_j\,,\end{aligned}$$

where the last equality comes from the solution to Problem 3.4. Then the expressions for cumulants are easily obtained.

4.5 We have

$$\begin{aligned}\phi_{1T}(\theta;x) &= \left|\begin{pmatrix} I_T-\dfrac{2i\theta x}{T^2}C'C & \dfrac{i\theta}{T^2}C'C \\ \dfrac{i\theta}{T^2}C'C & I_T\end{pmatrix}\right|^{-\frac{1}{2}} \\ &= \left|I_T-\frac{2i\theta x}{T^2}C'C+\frac{\theta^2}{T^4}(C'C)^2\right|^{-\frac{1}{2}} \\ &= \prod_{t=1}^{T}\left(1-\frac{2i\theta x}{T^2}\lambda_t-\frac{(i\theta)^2}{T^4}\lambda_t^2\right)^{-\frac{1}{2}}\,,\end{aligned}$$

which yields (1.67).

4.6 We first note that

$$X_{2T} = \underset{\sim}{\varepsilon}'\begin{pmatrix}\dfrac{x}{T^2}C'MC & -\dfrac{1}{2T^2}C'MC \\ -\dfrac{1}{2T^2}C'MC & 0\end{pmatrix}\underset{\sim}{\varepsilon}\,,$$

where $M = I_T - ee'/T$. Then we can proceed in the same way as in the solution to Problem 4.5 to arrive at (1.70).

4.7 From (1.39) we have

$$\text{(S2)} \qquad \mu_j(k) = \frac{1}{(k-1)!}\int_0^\infty \theta_2^{k-1}\frac{\partial^k \psi_j(\theta_1,-\theta_2)}{\partial\theta_1^k}\bigg|_{\theta_1=0} d\theta_2\,,$$

where

$$\begin{aligned}
\psi_1(\theta_1,-\theta_2) &= \phi_1(i\theta_1;\theta_2/\theta_1) \\
&= \left[\cosh\sqrt{\theta_2+\sqrt{\theta_1^2+\theta_2^2}}\;\cosh\sqrt{\theta_2-\sqrt{\theta_1^2+\theta_2^2}}\right]^{-\frac{1}{2}}, \\
\psi_2(\theta_1,-\theta_2) &= \phi_2(i\theta_1;\theta_2/\theta_1) \\
&= \left[\frac{\sinh\sqrt{\theta_2+\sqrt{\theta_1^2+\theta_2^2}}}{\sqrt{\theta_2+\sqrt{\theta_1^2+\theta_2^2}}}\;\frac{\sinh\sqrt{\theta_2-\sqrt{\theta_1^2+\theta_2^2}}}{\sqrt{\theta_2-\sqrt{\theta_1^2+\theta_2^2}}}\right]^{-\frac{1}{2}}.
\end{aligned}$$

Then we have, for instance,

$$\begin{aligned}
\mu_1(2) &= \frac{1}{4}\int_0^\infty \left\{(\cosh\sqrt{2\theta})^{-\frac{1}{2}} - (\cosh\sqrt{2\theta})^{-\frac{3}{2}}\;\frac{\sinh\sqrt{2\theta}}{\sqrt{2\theta}}\right\} d\theta \\
&= \frac{1}{4}\int_0^\infty \frac{u}{\sqrt{\cosh u}}du - \frac{1}{2}\,.
\end{aligned}$$

The other moments can be derived similarly. We used computerized algebra REDUCE to differentiate $\psi_j(\theta_1,-\theta_2)$.

4.8 We have only to show that $F_j(-x) = 1 - F_j(x)$. Because of the definition of $\phi_j(\theta;x)$ it is easy to see that $\phi_j(\theta;-x) = \phi_j(-\theta;x)$ so that

$$\begin{aligned}
F_j(-x) &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{1}{\theta}\,\mathrm{Im}(\phi_j(-\theta;x))d\theta \\
&= \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{1}{\theta}\,\mathrm{Im}(\phi_j(\theta;x))d\theta \\
&= 1 - F_j(x)\,.
\end{aligned}$$

5.1 It is easy to see that

$$\phi_{1T}(\theta;x) = \left|\begin{pmatrix} I_T - \dfrac{2i\theta x}{T^2}C'C & \dfrac{i\theta}{T}C' \\ \dfrac{i\theta}{T}C & I_T \end{pmatrix}\right|^{-\frac{1}{2}}$$

$$
\begin{aligned}
&= \left| I_T - \frac{2i\theta x}{T^2} C'C + \frac{\theta^2}{T^2} C'C \right|^{-\frac{1}{2}} \\
&= \prod_{t=1}^{T} \left[1 - (2i\theta x - \theta^2) \frac{\lambda_t}{T^2} \right]^{-\frac{1}{2}},
\end{aligned}
$$

and this last expression can be further factored as in (1.82).

5.2 We have

$$
X_{2T} = \underset{\sim}{\varepsilon}' \begin{pmatrix} \dfrac{x}{T^2} C'MC & -\dfrac{1}{2T} C'M \\ -\dfrac{1}{2T} MC & 0 \end{pmatrix} \underset{\sim}{\varepsilon},
$$

where $M = I_T - ee'/T$. Then we can proceed in the same way as in the solution to Problem 5.1 to arrive at (1.85).

5.3 In (S2) we have

$$
\begin{aligned}
\psi_1(\theta_1, -\theta_2) &= \left(\cosh \sqrt{2\theta_2 - \theta_1^2} \right)^{-\frac{1}{2}}, \\
\psi_2(\theta_1, -\theta_2) &= \left(\frac{\sinh \sqrt{2\theta_2 - \theta_1^2}}{\sqrt{2\theta_2 - \theta_1^2}} \right)^{-\frac{1}{2}}.
\end{aligned}
$$

Then we can arrive at (1.87) after some manipulations. We used computerized algebra REDUCE to differentiate $\psi_j(\theta_1, -\theta_2)$.

5.4 We have only to show that $F_j(-x) = 1 - F_j(x)$, which can be easily checked as in the solution to Problem 4.8.

Chapter 2.

1.1 Suppose that $\text{l.i.m.}\, X_n = X$ and $\text{l.i.m.}\, X_n = Y$. Since

$$
\begin{aligned}
E\left[(X-Y)^2\right] \le\ & E\left[(X_n - X)^2\right] + 2\sqrt{E\left[(X_n - X)^2\right] E\left[(X_n - Y)^2\right]} \\
& + E\left[(X_n - Y)^2\right]
\end{aligned}
$$

and the right side converges to 0, $E\left[(X-Y)^2\right] = 0$. Thus we have $P(X = Y) = 1$.

1.2 Since $E[(X_m - X_n)^2] = E(X_m^2 - 2X_mX_n + X_n^2) = 2\left(1 - \dfrac{1}{\sqrt{mn}}\right)$ for $m \neq n$, which does not tend to 0, $\{X_n\}$ does not converge in the m.s. sense. For any $\varepsilon > 0$, we have $P(|X_n| > \varepsilon) = P(X_n = \sqrt{n}) = \dfrac{1}{n} \to 0$; hence $\{X_n\}$ converges in probability to 0.

1.3 Put $Z_n(t) = aX_n(t) + bY_n(t)$ and $Z(t) = aX(t) + bY(t)$. Note that $\text{l.i.m.}\, Z_n(t) = Z(t)$. For any $q \times 1$ vector c we have

$$|E[c'(Z_n(t) - Z(t))]| \leq \sqrt{c'cE[(Z_n(t) - Z(t))'(Z_n(t) - Z(t))]}$$

so that $E(Z_n(t)) \to E(Z(t))$ as $n \to \infty$. Consider next

$$\begin{aligned} X_n'(t)Y_n(t) - X'(t)Y(t) &= (X_n(t) - X(t))'(Y_n(t) - Y(t)) \\ &\quad + X'(t)(Y_n(t) - Y(t)) + (X_n(t) - X(t))'\,Y(t)\,. \end{aligned}$$

Taking expectations leads from the Cauchy-Schwarz inequality to $E(X_n'(t)Y_n(t)) \to E(X'(t)Y(t))$.

1.4 $E(Y_m(t)Y_n(t)) \to 2$ as $m = n \to \infty$, while it converges to 1 as $m, n(\neq m) \to \infty$. Thus $\{Y_n(t)\}$ does not converge in the m.s. sense.

1.5 From Theorem 2.2 it holds that $Y(t)$ is m.s. continuous if and only if $E(Y'(t + h_1)\,Y(t + h_2))$ converges to $E(Y'(t)Y(t))$ as $h_1, h_2 \to 0$ in any manner, which is equivalent to the condition that $E(Y'(s)Y(t))$ is continuous at (t, t).

1.6 Since $\{Y(t)\}$ is m.s. continuous at every $t \in [a, b]$,

$$\underset{h_1 \to 0}{\text{l.i.m.}}\, Y(s + h_1) = Y(s)\,, \quad \underset{h_2 \to 0}{\text{l.i.m.}}\, Y(t + h_2) = Y(t)\,.$$

Therefore it follows from Theorem 2.1 that $E(Y'(s + h_1)Y(t + h_2)) \to E(Y'(s)Y(t))$ as $h_1,\ h_2 \to 0$.

1.7 Noting that $E(X(t)) = V(X(t)) = \lambda t$, we have, for $s < t$,

$$\begin{aligned} E(X(s)X(t)) &= E(X(s))E(X(t) - X(s)) + E\left(X^2(s)\right) \\ &= \lambda s + \lambda^2 st \end{aligned}$$

so that $E(X(s)X(t)) = \lambda \min(s,t) + \lambda^2 st$, which is continuous at every (t,t). Thus $\{X(t)\}$ is m.s. continuous at all t. On the other hand it holds that

$$\begin{aligned} &E[\,(\,X(t+h) - X(t)\,)\,(\,X(t+k) - X(t)\,)\,]\,/\,(hk) \\ &\quad = \lambda\,\frac{\min(h,k) - \min(h,0) - \min(0,k)}{hk} + \lambda^2\,, \end{aligned}$$

which does not converge as $h,\ k \to 0$. Thus $\{X(t)\}$ is nowhere m.s. differentiable.

1.8 It follows from Theorem 2.1 that

$$\begin{aligned} E(\dot{Y}(t)) &= E\left(\operatorname*{l.i.m.}_{h\to 0}\frac{Y(t+h)-Y(t)}{h}\right) \\ &= \lim_{h\to 0} E\left(\frac{Y(t+h)-Y(t)}{h}\right) = \frac{d}{dt}E(Y(t))\,. \end{aligned}$$

The relation (2.3) can be proved similarly.

1.9 Note first that $E(Y(t)Y(t+h)) = \dfrac{1}{2}\cos\omega h$ so that

$$\begin{aligned} &E\left[(Y(t+h_1) - Y(t))(Y(t+h_2) - Y(t))\right]/(h_1h_2) \\ &\quad = \left[\cos\omega(h_2-h_1) - \cos\omega h_1 - \cos\omega h_2 + 1\right]/(2h_1h_2) \longrightarrow \frac{\omega^2}{2}\,. \end{aligned}$$

2.1 (a) $E[(w(t+h)-w(t))'(w(t+h)-w(t))] = q|h| \to 0$ as $h \to 0$.

(b) $E[(w(t+h)-w(t))'(w(t+h)-w(t))]/h^2 = q/|h|$ does not converge as $h \to 0$.

(c) Put $\Delta w_i = w(t_i) - w(t_{i-1})$ and $\Delta t_i = t_i - t_{i-1}$. Then

$$\begin{aligned} E\left[\left(\sum_{i=1}^n \Delta w_i'\Delta w_i - (b-a)q\right)^2\right] &= E\left[\left\{\sum_{i=1}^n (\Delta w_i'\Delta w_i - q\Delta t_i)\right\}^2\right] \\ = \sum_{i=1}^n E\left[(\Delta w_i'\Delta w_i - q\Delta t_i)^2\right] &= 2q\sum_{i=1}^n (\Delta t_i)^2 \le 2q\Delta_n(b-a) \to 0\,. \end{aligned}$$

2.2 It is clear that $w(0) \equiv 0$ and $E(w(t)) = E(\Delta w_i) = 0$, where $\Delta w_i = w(t_i) - w(t_{i-1})$. For $t_{i-1} < t_i \le t_{k-1} < t_k$ we have

$$E(\Delta w_i \Delta w_k') = \sum_{n=1}^{\infty} \frac{2}{\left(\left(n-\dfrac{1}{2}\right)\pi\right)^2}[\sin a_{ni} - \sin a_{n,i-1}][\sin a_{nk} - \sin a_{n,k-1}]I_q$$

$$= \sum_{n=1}^{\infty} \frac{1}{\left(\left(n-\frac{1}{2}\right)\pi\right)^2}[\cos(a_{ni}-a_{nk}) - \cos(a_{ni}+a_{nk}) - \cos(a_{ni}-a_{n,k-1})$$
$$+\cos(a_{ni}+a_{n,k-1}) - \cos(a_{n,i-1}-a_{n,k}) + \cos(a_{n,i-1}+a_{n,k})$$
$$+\cos(a_{n,i-1}-a_{n,k-1}) - \cos(a_{n,i-1}+a_{n,k-1})]I_q\,,$$

where $a_{ni} = \left(n-\frac{1}{2}\right)\pi t_i$. Using the formula given in the problem it can be shown that $E(\Delta w_i \Delta w_k') = 0$. Similarly we have $E(\Delta w_i \Delta w_i') = (t_i - t_{i-1})I_q$ so that Δw_i is independent $\mathrm{N}(0, (t_i - t_{i-1})I_q)$. We also have $w(t) \sim \mathrm{N}(0, tI_q)$. Thus $\{w(t)\}$ is the q-dimensional standard Brownian motion.

2.3 For $s \le t$ we have

$$\begin{pmatrix} w(s) \\ w(t) \\ w(1) \end{pmatrix} \sim \mathrm{N}\left(0, \Sigma\right)\,, \qquad \Sigma = \begin{pmatrix} s & s & s \\ s & t & t \\ s & t & 1 \end{pmatrix} \otimes I_q$$

with $\otimes$ being the Kronecker product. Then it holds that $E(w(t)|w(1)=0) = 0$ and

$$V(w(s), w(t)\mid w(1)=0) = \begin{pmatrix} s-s^2 & s-st \\ s-st & t-t^2 \end{pmatrix} \otimes I_q$$

so that $\mathrm{Cov}(w(s), w(t)\mid w(1)=0) = (\min(s,t) - st)\, I_q\,.$

2.4 It is clear that $\{\bar{w}(t)\}$ is a Gaussian process with $\bar{w}(0) = \bar{w}(1) \equiv 0$ and $E(\bar{w}(t)) = 0$. Moreover, for $s < t$, we have

$$\begin{aligned} E(\bar{w}(s)\bar{w}'(t)) &= \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2}(\cos n\pi(s-t) - \cos n\pi(s+t))\, I_q \\ &= \left[\frac{1}{4}(t-s-1)^2 - \frac{1}{12} - \left(\frac{1}{4}(s+t-1)^2 - \frac{1}{12}\right)\right] I_q \\ &= (s-st)\, I_q\,. \end{aligned}$$

3.1 From Theorem 2.2 the integral in (2.7) exists if and only if $E(V_m'(t)V_n(t))$ converges to a finite function on $[a,b]$ as $m,n \to \infty$ in any manner, which is equivalent to the condition that the integral in (2.8) exists and is finite.

3.2 It follows from the solution to Problem 1.5 that $E(Y'(s)Y(t))$ is continuous on $[a,b]\times[a,b]$. Then it is clear that the double Riemann integral $\iint E(Y'(s)Y(t))dsdt$ exists and is finite. Thus $\{Y(t)\}$ is m.s. integrable by Theorem 2.3.

3.3 Since $E(w(s)w(t)) = \min(s,t)$, it is easy to obtain $E(V) = \frac{1}{2}$ and $E(W) = \frac{1}{6}$. Noting also that $E\left(w^2(s)w^2(t)\right) = 2\min^2(s,t) + st$ we obtain $E(V^2) = \frac{7}{12}$ and $E(W^2) = \frac{1}{20}$.

3.4 From Theorem 2.3 we have only to check that the integral (2.8) with $f(r,t)$ and $Y(r)$ replaced by $I_{[0,t]}(r)$ and $w(r)$, respectively, exists and is finite. The integral is

$$\int_0^1\int_0^1 I_{[0,t]}(r)\, I_{[0,t]}(s)\, E(w'(r)w(s))drds = \frac{qt^3}{3}$$

so that $V(t)$ in (2.10) is well defined. From Theorem 2.4 and the above result we have that $V(t) \sim \mathrm{N}\left(0, t^3 I_q/3\right)$.

3.5 Let us define

$$V_{1m} = \sum_{i=1}^{m}(1-s_i')(w(s_i)-w(s_{i-1}))$$

and consider

$$E(V_{1m}'V_{1n}) = \sum_{i=1}^{m}\sum_{j=1}^{n}(1-s_i')(1-t_j')E\left[(w(s_i)-w(s_{i-1}))'\,(w(t_j)-w(t_{j-1}))\right].$$

It can be checked that this last quantity converges to $q\int_0^1(1-t)^2dt = q/3$ as $m,\ n\to\infty$. Thus V_1 is well defined.

3.6 We may assume that H is diagonal and thus we have only to show that

$$A = \int_0^1\int_0^1\int_0^1\int_0^1 K(s,t)K(u,v)E(dw(s)dw(t)dw(u)dw(v))$$

exists and is finite when $\{w(t)\}$ is scalar. We have

$$E(dw(s)dw(t)dw(u)dw(v)) = \begin{cases} 3(dt)^2 & s=t=u=v \\ ds\,du & s=t,\ u=v,\ s\neq u \\ ds\,dt & s=u,\ t=v,\ s\neq t \\ ds\,dt & s=v,\ t=u,\ s\neq t \\ 0 & \text{otherwise} \end{cases}$$

so that

$$
\begin{aligned}
A &= 3\int_0^1 K^2(s,t)(dt)^2 + \underset{s\neq t}{SS}\, K(s,s)K(t,t)dsdt + 2\,\underset{s\neq t}{SS}\, K^2(s,t)dsdt \\
&= \left(\int_0^1 K(s,s)ds\right)^2 + 2\int_0^1\int_0^1 K^2(s,t)dsdt < \infty\,.
\end{aligned}
$$

3.7 As for (2.22) we evaluate $\lim_{m\to\infty} E(X_{m,m})$, where $X_{m,m}$ is defined in (2.21) with $s_i = t_i$ and $s_i' = t_i'$. Putting $\Delta w_i = w(s_i) - w(s_{i-1})$ and $\Delta s_i = s_i - s_{i-1}$, we have

$$
E(X_{m,m}) = \sum_{i=1}^m K(s_i', s_i')\Delta s_i \,\mathrm{tr}(H) \longrightarrow E(X)\,.
$$

For (2.23) we consider

$$
E\left(X_{m,m}^2\right) = \sum_{i=1}^m\sum_{j=1}^m\sum_{k=1}^m\sum_{l=1}^m K(s_i', s_j')K(s_k', s_l')E[\Delta w_i' H \Delta w_j \Delta w_k' H \Delta w_l]\,,
$$

where $E[\quad]$ is equal to

$$
\begin{aligned}
&\sum_{a=1}^q H_{aa}^2\, E(\Delta w_{ai}\Delta w_{aj}\Delta w_{ak}\Delta w_{al}) + \sum_{a\neq b} H_{aa}H_{bb}\, E(\Delta w_{ai}\Delta w_{aj}\Delta w_{bk}\Delta w_{bl}) \\
&+ \sum_{a\neq b} H_{ab}^2\, E(\Delta w_{ai}\Delta w_{bj}\Delta w_{ak}\Delta w_{bl}) + \sum_{a\neq b} H_{ab}^2\, E(\Delta w_{ai}\Delta w_{bj}\Delta w_{bk}\Delta w_{al}).
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
E(X_{m,m}^2) &= \sum_{a=1}^q H_{aa}^2 \left[3\sum_{i=1}^m K^2(s_i', s_i')(\Delta s_i)^2 + \sum_{i\neq j} K(s_i', s_i')K(s_j', s_j')\Delta s_i \Delta s_j\right. \\
&\quad \left. + 2\sum_{i\neq j} K^2(s_i', s_j')\Delta s_i\Delta s_j\right] + \sum_{a\neq b} H_{aa}H_{bb}\left(\sum_{i=1}^m K(s_i', s_i')\Delta s_i\right)^2 \\
&\quad + 2\sum_{a\neq b} H_{ab}^2 \sum_{i=1}^m\sum_{j=1}^m K^2(s_i', s_j')\Delta s_i \Delta s_j \\
&= \left(\sum_{a=1}^q H_{aa}\right)^2 \left(\sum_{i=1}^m K(s_i', s_i')\Delta s_i\right)^2 \\
&\quad + 2\sum_{a=1}^q\sum_{b=1}^q H_{ab}^2 \sum_{i=1}^m\sum_{j=1}^m K^2(s_i', s_j')\Delta s_i\Delta s_j\,,
\end{aligned}
$$

which converges to $E\left(X^2\right)$ given in (2.23).

3.8 It is easy to see that the left side is equal to

$$\mathcal{L}\left(\sum_{i=1}^{q}\lambda_i\int_0^1\int_0^1 g(s)g(t)dw_i(s)dw_i(t)\right)=\mathcal{L}\left(\sum_{i=1}^{q}\lambda_i\left(\int_0^1 g(t)dw_i(t)\right)^2\right).$$

Since $\int_0^1 g(t)dw_i(t)\ (i=1,\cdots,q)\sim \mathrm{NID}\left(0,\int_0^1 g^2(t)dt\right)$, we have the conclusion.

3.9 As for (2.27) we have

$$\begin{aligned}\int_0^1 \bar{w}'(t)H\bar{w}(t)dt &= \int_0^1 [w(t)-tw(1)]'\,H[w(t)-tw(1)]dt\\ &= \int_0^1 [w'(t)Hw(t)-tw'(t)Hw(1)-w'(1)Htw(t)\\ &\qquad +w'(1)Hw(1)t^2]dt\,,\end{aligned}$$

where

$$\begin{aligned}\int_0^1 w'(t)Hw(t)dt &= \int_0^1\left(\int_0^t\int_0^t dw'(u)Hdw(v)\right)dt\\ &= \int_0^1\int_0^1[1-\max(s,t)]dw'(s)Hdw(t)\,,\end{aligned}$$

$$\int_0^1 tw'(t)Hw(1)dt=\int_0^1\int_0^1\frac{1-s^2}{2}dw'(s)Hdw(t)\,,$$

$$\int_0^1 w'(1)Hw(1)t^2dt=\frac{1}{3}\int_0^1\int_0^1 dw'(s)Hdw(t)\,.$$

Substituting these into the right side above we obtain the left side of (2.27). The relation (2.26) can be proved similarly.

4.1 We prove (2.31) by induction on g. When $g=0$, it clearly holds that $F_0(t)=w(t)$. Suppose that (2.31) holds for $g=k-1$. Then we have

$$\begin{aligned}F_k(t) &= \int_0^t F_{k-1}(s)ds=\int_0^t\left(\int_0^s\frac{(s-u)^{k-1}}{(k-1)!}dw(u)\right)ds\\ &= \int_0^t\left(\int_u^t\frac{(s-u)^{k-1}}{(k-1)!}ds\right)dw(u)=\int_0^t\frac{(t-u)^k}{k!}dw(u)\,,\end{aligned}$$

which establishes (2.31).

4.2 From (2.29) and (2.31) we have

$$\begin{aligned}\int_0^1 \bar{F}_g(t)dt &= \int_0^1 [F_g(t) - tF_g(1)]dt \\ &= \int_0^1 \left(\int_0^t \frac{(t-s)^g}{g!} dw(s)\right) dt - \frac{1}{2}\int_0^1 \frac{(1-s)^g}{g!} dw(s) \\ &= \int_0^1 \left[\int_s^1 \frac{(t-s)^g}{g!} dt - \frac{(1-s)^g}{2(g!)}\right] dw(s) \\ &= \int_0^1 \left[\frac{(1-s)^{g+1}}{(g+1)!} - \frac{(1-s)^g}{2(g!)}\right] dw(s)\,.\end{aligned}$$

We also have, from (2.30) and (2.31),

$$\begin{aligned}\int_0^1 \tilde{F}_g'(t)\tilde{F}_g(t)dt &= \int_0^1 F_g'(t)F_g(t)dt - \int_0^1\int_0^1 F_g'(s)F_g(t)dsdt \\ &= \int_0^1 \left(\int_0^t\int_0^t \frac{((t-u)(t-v))^g}{(g!)^2} dw'(u)dw(v)\right) dt \\ &\qquad - \int_0^1\int_0^1 \frac{((1-s)(1-t))^{g+1}}{((g+1)!)^2} dw'(s)dw(t) \\ &= \int_0^1\int_0^1 \left[K_g(s,t) - \frac{((1-s)(1-t))^{g+1}}{((g+1)!)^2}\right] dw'(s)dw(t)\,.\end{aligned}$$

4.3 Since $F_g(t)$ is continuously differentiable, we have

$$\begin{aligned}\int_0^t F_g(s)dF_g'(s) &= F_g(s)F_g'(s)\Big|_0^t - \left(\int_0^t F_g(s)dF_g'(s)\right)' \\ &= F_g(t)F_g'(t) - \left(\int_0^t F_g(s)dF_g'(s)\right)',\end{aligned}$$

which proves (2.33).

5.1 Put $\tau_{i-1} = (1-a)s_{i-1} + as_i$ and $\Delta w_i = w(s_i) - w(s_{i-1})$. Then we have

$$\sum_{i=1}^m w(\tau_{i-1})\Delta w_i = \sum_{i=1}^m w(s_{i-1})\Delta w_i + \sum_{i=1}^m (w(\tau_{i-1}) - w(s_{i-1}))\Delta w_i\,.$$

Here the first term on the right side converges in m.s. to $\frac{1}{2}(w^2(t)-t)$, while the second term can be rewritten as

$$\text{(S3)}\qquad \sum_{i=1}^m (w(\tau_{i-1}) - w(s_{i-1}))^2 + \sum_{i=1}^m (w(\tau_{i-1}) - w(s_{i-1}))(w(s_i) - w(\tau_{i-1}))\,.$$

Since

$$
\begin{aligned}
E\left[\sum_{i=1}^{m}(w(\tau_{i-1})-w(s_{i-1}))^2\right] &= \sum_{i=1}^{m}(\tau_{i-1}-s_{i-1}) = a\sum_{i=1}^{m}(s_i-s_{i-1}) = at\,,\\
V\left[\sum_{i=1}^{m}(w(\tau_{i-1})-w(s_{i-1}))^2\right] &= 2\sum_{i=1}^{m}(\tau_{i-1}-s_{i-1})^2 \le 2a^2t\,\max_i(s_i-s_{i-1})\,,
\end{aligned}
$$

the first term in (S3) converges in m.s. to at, while the second term in (S3) can be shown to converge in m.s. to 0, which establishes (2.40).

5.2 We have already shown that $U(t)$ is m.s. continuous. As for the martingale property we first have, for $s \le t$, $E(U(t)|U(s)) = U(s) + E(U(t)-U(s)\,|\,U(s))$. Noting that

$$
\begin{aligned}
U(t)-U(s) &= \int_s^t X(u)dw(u)\\
&= \underset{\substack{m\to\infty\\ \Delta_m\to 0}}{\text{l.i.m.}} \sum_{i=1}^{m} X(u_{i-1})(w(u_i)-w(u_{i-1}))\,,
\end{aligned}
$$

where $s = u_0 < u_1 < \cdots < u_m = t$, we can deduce that

$$
\begin{aligned}
&E[X(u_{i-1})(w(u_i)-w(u_{i-1}))\,|\,U(s)]\\
&\quad = E[E[X(u_{i-1})(w(u_i)-w(u_{i-1}))\,|\,U(u_{i-1})]\,|\,U(s)]\\
&\quad = E[X(u_{i-1})E[w(u_i)-w(u_{i-1})\,|\,U(u_{i-1})]\,|\,U(s)]\\
&\quad = 0.
\end{aligned}
$$

Then it is seen that $E(U(t)-U(s)|U(s)) = 0$ so that $U(t)$ is a martingale.

5.3 Putting $\Delta w_i = w(s_i)-w(s_{i-1})$ and $\Delta s_i = s_i - s_{i-1}$ we have

$$
\begin{aligned}
&E\left[\left\{\sum_{i=1}^{m} X(s_{i-1})\left((\Delta w_i)^2-\Delta s_i\right)\right\}^2\right]\\
&\quad = E\left[\sum_{i=1}^{m}\sum_{j=1}^{m} X(s_{i-1})X(s_{j-1})\left((\Delta w_i)^2-\Delta s_i\right)\left((\Delta w_j)^2-\Delta s_j\right)\right]\\
&\quad = 2\sum_{i=1}^{m} E\left(X^2(s_{i-1})\right)(\Delta s_i)^2 \le 2\max_i E\left(X^2(s_{i-1})\right)\max_i \Delta s_i \sum_{i=1}^{m}\Delta s_i\\
&\quad \longrightarrow 0
\end{aligned}
$$

as $m \to \infty$ and $\max_i \Delta s_i \to 0$ so that (2.41) is established. The relation (2.42) can be proved similarly.

5.4 Because of the definition of y_t we have

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T} y_{t-1}\varepsilon_t &= \frac{1}{T}\sum_{t=1}^{T} y_{t-1}(y_t - y_{t-1}) \\
&= -\frac{1}{2T}\left[\sum_{t=1}^{T}(y_t - y_{t-1})^2 - \sum_{t=1}^{T} y_t^2 + \sum_{t=1}^{T} y_{t-1}^2\right] \\
&= \frac{1}{2T}y_T^2 - \frac{1}{2T}\sum_{t=1}^{T}\varepsilon_t^2 \\
&= \frac{1}{2}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_t\right)^2 - \frac{1}{2T}\sum_{t=1}^{T}\varepsilon_t^2 ,
\end{aligned}$$

which converges in distribution to $\frac{1}{2}(w^2(1) - 1)$ by the central limit theorem and the law of large numbers.

6.1 Note first that the c.f. $\phi(\theta)$ of the right hand side in (2.46) is given by $\phi(\theta) = (\cosh\theta)^{-\frac{1}{2}}$. Thus it suffices to show that the c.f. $\phi_m(\theta)$ of

$$V_m(1) = \sum_{j=1}^{m} w_a(t_{j-1})(w_b(t_j) - w_b(t_{j-1}))\,, \qquad (a \neq b)$$

converges to $\phi(\theta)$ as $m \to \infty$, where $t_j = j/m$. It is easy to check that $V_m(1)$ has the same limiting distribution as U_m, where U_m is given in (1.48). Thus we have the conclusion from the arguments there.

6.2 It is easy to obtain

$$\begin{aligned}
&\sum_{i=1}^{m} w(t_{i-1})(w(t_i) - w(t_{i-1}))' + \sum_{i=1}^{m}(w(t_i) - w(t_{i-1}))w'(t_{i-1}) \\
&\quad = w(t)w'(t) - \sum_{i=1}^{m}(w(t_i) - w(t_{i-1}))(w(t_i) - w(t_{i-1}))' ,
\end{aligned}$$

where $0 = t_0 < t_1 < \cdots < t_m = t$. Letting $m \to \infty$ and $\Delta_m = \max_i(t_i - t_{i-1}) \to 0$, (2.47) is established by the law of large numbers.

6.3 Premultiplying A on both sides of (2.47) with $t=1$ and taking the trace lead from $\mathrm{tr}(A)=0$ to

$$\int_0^1 w'(s)Adw(s) = \frac{1}{2}w'(1)Aw(1)\,.$$

Then we have $\mathcal{L}(w'(1)Aw(1)/2) = \mathcal{L}\left(\sum_{i=1}^{q}\lambda_i Z_i^2\right)$, where λ_i's are the eigenvalues of $A/2$ and $\{Z_i\} \sim \mathrm{NID}(0,1)$.

7.1 To establish (2.55) we use Ito's theorem putting $f(x,t)=x^n$ with $dX(t)$ defined in (2.52). As for (2.56) we put $f(x,t)=x^n$ with $dX(t)=dw(t)$ so that $\mu=0$ and $\sigma=1$. To prove (2.57) we put $f(x,t)=e^x$ with $dX(t)=dw(t)$, $\mu=0$ and $\sigma=1$.

7.2 Since the existence and uniqueness of the solution to (2.60) is ensured, we have only to show that $d\left(X(0)e^{w(t)-\frac{t}{2}}\right) = X(t)dw(t)$, which is almost trivial.

7.3 We have, by Ito's theorem,

$$\begin{aligned} dX(t) &= \left(\alpha e^{\alpha t}X(0) + \beta e^{\alpha t} + \gamma\alpha e^{\alpha t}\int_0^t e^{-\alpha s}dw(s)\right)dt + \gamma dw(t) \\ &= (\alpha X(t)+\beta)dt + \gamma dw(t)\,, \end{aligned}$$

which gives (2.61).

7.4 For $s \le t$ we have, from (2.63),

$$\begin{aligned} \mathrm{Cov}(X(s),X(t)) &= e^{\alpha(s+t)}\mathrm{Cov}\left(X(0)+\int_0^s e^{-\alpha u}dw(u),\, X(0)+\int_0^t e^{-\alpha v}dw(v)\right) \\ &= e^{\alpha(s+t)}\left[V(X(0)) + \int_0^s e^{-2\alpha u}du\right] \\ &= e^{\alpha(s+t)}\left[V(X(0)) + \frac{1-e^{-2\alpha s}}{2\alpha}\right], \end{aligned}$$

which establishes (2.64).

7.5 From the definition of X(t) in (2.63) we have

$$\int_0^1 e^{2\alpha t}\left[X^2(0) + 2X(0)\int_0^t e^{-\alpha s}dw(s) + \int_0^t\!\!\int_0^t e^{-\alpha(u+v)}dw(u)dw(v)\right]dt$$

$$
\begin{aligned}
&= \frac{e^{2\alpha}-1}{2\alpha}X^2(0) + 2X(0)\int_0^1 \left(\int_s^1 e^{2\alpha t}dt\right) e^{-\alpha s}dw(s) \\
&\qquad + \int_0^1\int_0^1 \left(\int_{\max(u,v)}^1 e^{2\alpha t}dt\right) e^{-\alpha(u+v)}dw(u)dw(v) \\
&= \frac{e^{2\alpha}-1}{2\alpha}X^2(0) + X(0)\int_0^1 \frac{e^{\alpha(2-s)}-e^{\alpha s}}{\alpha}\,dw(s) \\
&\qquad + \int_0^1\int_0^1 \frac{e^{\alpha(2-u-v)}-e^{\alpha|u-v|}}{2\alpha}dw(u)dw(v)\,.
\end{aligned}
$$

7.6 As for (2.72) we put $Y_1(t) = e^{\alpha t}w(t)$ and

$$
Y_2(t) = \int_0^t e^{-\alpha s}dw(s)
$$

so that $dY_1(t) = e^{\alpha t}(\alpha w(t)dt + dw(t))$ and $dY_2(t) = e^{-\alpha t}dw(t)$. Then (2.71) yields (2.72). As for (2.73) we put $Y_1(t) = w(t)$ while $Y_2(t)$ is the same as above. Thus $dY_1(t) = dw(t)$ and $dY_2(t) = e^{-\alpha t}dw(t)$. Define $g(y,t) = \exp(y_1y_2)$ so that $g_t = 0$ and

$$
g_y = \begin{pmatrix} y_2 \\ y_1 \end{pmatrix} g\,, \qquad g_{yy} = \begin{pmatrix} y_2^2 & 1+y_1y_2 \\ 1+y_1y_2 & y_1^2 \end{pmatrix} g\,.
$$

Then (2.70) yields

$$
dg = \left\{Y_2dY_1 + Y_1dY_2 + \frac{1}{2}\left(Y_2^2 + 2(1+Y_1Y_2)e^{-\alpha t} + Y_1^2e^{-2\alpha t}\right)dt\right\}g\,,
$$

which leads us to (2.73).

Chapter 3.

1.1 Since $\rho(x,y)$ is a metric, we have

$$
|\rho(x,y) - \rho(\tilde{x},\tilde{y})| \le \rho(x,\tilde{x}) + \rho(y,\tilde{y})
$$

which can be proved by the triangle inequalities :

$$
\rho(x,y) \le \rho(x,\tilde{x}) + \rho(\tilde{x},\tilde{y}) + \rho(\tilde{y},y)\,, \quad \rho(\tilde{x},\tilde{y}) \le \rho(\tilde{x},x) + \rho(x,y) + \rho(y,\tilde{y})\,.
$$

Then it is clear that $\rho(x,y)$ is a continuous function of x and y.

1.2 Let $\{x_n\}$ be a fundamental sequence in C, that is, $\rho(x_m, x_n) \to 0$ as m, $n \to \infty$. Because of the definition of ρ and completeness of the real line, $\{x_n(t)\}$ converges uniformly in t so that the limit $x(t)$ lies in C and $\rho(x_n, x) \to 0$. Thus C is complete. Separability follows from the Weierstrass approximation theorem which ensures that any x in C can be uniformly approximated by a polynomial with real coefficients, which, in turn, can be approximated by a polynomial with coefficients of rational numbers.

4.1 We have only to show that $E[\exp\{i\theta h(X_n)\}] \to E[\exp\{i\theta h(X)\}]$, where

$$E[\exp\{i\theta h(X_n)\}] = E\{\cos\theta h(X_n)\} + iE\{\sin\theta h(X_n)\}\,.$$

Since $f_1(X_n) = \cos\theta h(X_n)$ and $f_2(X_n) = \sin\theta h(X_n)$ are both bounded and continuous, it must hold that $E(f_1(X_n)) \to E(f_1(X))$ and $E(f_2(X_n)) \to E(f_2(X))$, from which the conclusion follows.

4.2 Suppose that $\rho(x, y) < \varepsilon$ so that $y(t) - \varepsilon < x(t) < y(t) + \varepsilon$ for all $t \in [0, 1]$. Then it follows that $|\sup x(t) - \sup y(t)| < \varepsilon$. To show that $h_2(x)$ is continuous we first have, by the triangle inequality,

$$\rho(x, 0) \le \rho(x, y) + \rho(y, 0)\,, \quad \rho(y, 0) \le \rho(y, x) + \rho(x, 0)$$

so that $|\rho(x, 0) - \rho(y, 0)| \le \rho(x, y)$, which means that

$$\left| \sup_{0\le t\le 1} |x(t)| - \sup_{0\le t\le 1} |y(t)| \right| \le \sup_{0\le t\le 1} |x(t) - y(t)|\,.$$

Thus $h_2(x)$ is shown to be continuous. The function $h_3(x)$ is the mapping which carries x of C to the point $(h_1(x),\ h_2(x))$ of R^2; so it is certainly continuous since $h_1(x)$ and $h_2(x)$ are both continuous.

4.3 Note that $P(\rho(X_n, c) < \varepsilon) = P(X_n \in N(c, \varepsilon))$, where $N(c, \varepsilon)$ is the open sphere with center c and radius ε. By the portmanteau theorem (Billingsley (1968, p.24)) $\mathcal{L}(X_n) \to \mathcal{L}(c)$ implies that $P(X_n \in N(c, \varepsilon)) \to P(c \in N(c, \varepsilon))$, which is certainly unity.

4.4 Given any $\varepsilon > 0$ there exists some $\delta > 0$ such that $|X_n - c| < \delta$ implies $|h(X_n) - h(c)| < \varepsilon$. Therefore we have

$$P(|h(X_n) - h(c)| \geq \varepsilon) \leq P(|X_n - c| \geq \delta)$$

so that, by assumption, $h(X_n) \to h(c)$ in probability.

4.5 Let x be a continuity point of $P(X < x)$. Then it holds that

$$\begin{aligned} P(Y_n < x) &= P(Y_n < x,\ X_n - Y_n < \varepsilon) + P(Y_n < x,\ X_n - Y_n \geq \varepsilon) \\ &\leq P(X_n < x + \varepsilon) + P(X_n - Y_n \geq \varepsilon) \end{aligned}$$

so that $\limsup_{n\to\infty} P(Y_n < x) \leq P(X < x + \varepsilon)$. We also have

$$\begin{aligned} P(X_n < x - \varepsilon) &= P(X_n < x - \varepsilon,\ Y_n < x) + P(X_n < x - \varepsilon,\ Y_n \geq x) \\ &\leq P(Y_n < x) + P(X_n - Y_n \leq -\varepsilon) \end{aligned}$$

so that $\liminf_{n\to\infty} P(Y_n < x) \geq P(X < x - \varepsilon)$. Since ε is arbitrary, we have $\mathcal{L}(Y_n) \to \mathcal{L}(X)$.

4.6 Noting that

$$\begin{aligned} P\left(\max_{1\leq j\leq T} \frac{|\varepsilon_j|}{\sqrt{T}} \leq \delta\right) &= \prod_{j=1}^{T} P\left(\frac{|\varepsilon_j|}{\sqrt{T}} \leq \delta\right) \\ &= \left\{1 - P\left(\frac{|\varepsilon_1|}{\sqrt{T}} > \delta\right)\right\}^T \\ &\geq \left[1 - \frac{1}{\delta^2 T} E\left\{\varepsilon_1^2 I\left(\frac{|\varepsilon_1|}{\sqrt{T}} > \delta\right)\right\}\right]^T, \end{aligned}$$

(3.11) is seen to hold.

4.7 For any fixed $c > 0$ we have, for any $\varepsilon > 0$,

$$\begin{aligned} P(|X_nY_n| > \varepsilon) &= P\left(|X_nY_n| > \varepsilon,\, |Y_n| \leq \frac{\varepsilon}{c}\right) + P\left(|X_nY_n| > \varepsilon,\, |Y_n| > \frac{\varepsilon}{c}\right) \\ &\leq P\left(|X_n| > c\right) + P\left(|Y_n| > \frac{\varepsilon}{c}\right). \end{aligned}$$

Thus $\limsup_{n\to\infty} P(|X_nY_n| > \varepsilon) \le P(|X| > c)$. Thus it is seen that $X_nY_n \to 0$ in probability by choosing c large (Rao (1973, p.122)).

4.8 Put $\tilde{X}_T(t) = X_T(t) - \sum_{j=1}^{T} X_T(j/T)/T$. Then $\mathcal{L}(\tilde{X}_T) \to \mathcal{L}(\tilde{w})$ by Corollary 3.2, where $\tilde{w}$ is the demeaned Brownian motion. We have

$$\begin{aligned}
&\left| \frac{1}{T^2}\sum_{j=1}^{T}(y_j - \bar{y})^2 - \int_0^1 \tilde{X}_T^2(t)dt \right| \\
&\qquad = \left| \sum_{j=1}^{T} \int_{\frac{j-1}{T}}^{\frac{j}{T}} \left(\tilde{X}_T^2\left(\frac{j}{T}\right) - \tilde{X}_T^2(t)\right) dt \right| \\
&\qquad \le 2 \sup_{0\le t\le 1} |\tilde{X}_T(t)| \max_{1\le j\le T} \frac{|\varepsilon_j|}{\sqrt{T}},
\end{aligned}$$

which converges in probability to 0. Thus the second relation in (3.13) is established by the continuous mapping theorem. Assuming that $\{\varepsilon_j\} \sim$ NID(0,1), we have

$$\begin{aligned}
\mathcal{L}\left(\frac{1}{T^2}\sum_{j=1}^{T}(y_j-\bar{y})^2\right) &= \mathcal{L}\left(\frac{1}{T^2}y'My\right) = \mathcal{L}\left(\frac{1}{T^2}\varepsilon' C'MC\varepsilon\right) \\
&= \mathcal{L}\left(\frac{1}{T^2}\varepsilon' MCC'M\varepsilon\right) = \mathcal{L}\left(\frac{1}{T^2}\sum_{j=1}^{T}\left(\sum_{i=j}^{T}(\varepsilon_i - \bar{\varepsilon})\right)^2\right) \\
&= \mathcal{L}\left(\frac{1}{T^2}\sum_{j=1}^{T}\left(\sum_{i=1}^{j}(\varepsilon_i - \bar{\varepsilon})\right)^2\right),
\end{aligned}$$

where $M = I_T - ee'/T$ with $e = (1, \cdots, 1)'$ and C is defined in (1.3). Put $\bar{X}_T(t) = X_T(t) - tX_T(1)$. Then $\mathcal{L}(\bar{X}_T) \to \mathcal{L}(\bar{w})$ by Corollary 3.1, where $\bar{w} = \{w(t) - tw(1)\}$. We now obtain

$$\begin{aligned}
&\left| \frac{1}{T^2}\sum_{j=1}^{T}\left(\sum_{i=1}^{j}(\varepsilon_i - \bar{\varepsilon})\right)^2 - \int_0^1 \bar{X}_T^2(t)dt \right| \\
&\qquad = \left| \sum_{j=1}^{T} \int_{\frac{j-1}{T}}^{\frac{j}{T}} \left(\bar{X}_T^2\left(\frac{j}{T}\right) - \bar{X}_T^2(t)\right) dt \right| \\
&\qquad \le 2 \sup_{0\le t\le 1} |\bar{X}_T(t)| \left(\max_{1\le j\le T} \frac{|\varepsilon_j|}{\sqrt{T}} + \frac{1}{T}|X_T(1)| \right),
\end{aligned}$$

which converges in probability to 0. Thus the last relation in (3.13) is established by the continuous mapping theorem.

4.9 It follows from (3.8) that

$$
\begin{aligned}
\hat{\sigma}^2 &= \frac{1}{T-1}\sum_{j=2}^{T}(y_j - y_{j-1} - (\hat{\rho}-1)y_{j-1})^2 \\
&= \frac{1}{T-1}\left[\sum_{j=2}^{T}\varepsilon_j^2 - 2(\hat{\rho}-1)\sum_{j=2}^{T}y_{j-1}\varepsilon_j + (\hat{\rho}-1)^2\sum_{j=2}^{T}y_{j-1}^2\right].
\end{aligned}
$$

Since $\hat{\rho}-1 = O_p(T^{-1})$ and

$$
\sum_{j=2}^{T}y_{j-1}\varepsilon_j = O_p(T)\,, \qquad \sum_{j=2}^{T}y_{j-1}^2 = O_p(T^2)\,,
$$

it is seen that $\hat{\sigma}^2 \to \sigma^2$ in probability.

5.1 There exists m such that $l|\alpha_l| < 1$ for all $l > m$. Thus we have

$$
\begin{aligned}
\sum_{l=0}^{\infty} l^2\alpha_l^2 &= \sum_{l=0}^{m} l^2\alpha_l^2 + \sum_{l=m+1}^{\infty} l^2\alpha_l^2 \\
&\le \sum_{l=0}^{m} l^2\alpha_l^2 + \sum_{l=m+1}^{\infty} l|\alpha_l| < \infty\,.
\end{aligned}
$$

5.2 Put $\alpha(L) = \sum_{l=0}^{\infty}\alpha_l L^l$, where L is the lag operator. Then $\alpha(L) = \alpha - (\alpha - \alpha(L))$, where

$$
\begin{aligned}
\alpha - \alpha(L) &= \sum_{k=1}^{\infty}\alpha_k\left(1 - L^k\right) = (1-L)\sum_{k=1}^{\infty}\alpha_k\sum_{l=0}^{k-1}L^l \\
&= (1-L)\sum_{l=0}^{\infty}\sum_{k=l+1}^{\infty}\alpha_k L^l = (1-L)\sum_{l=0}^{\infty}\tilde{\alpha}_l L^l
\end{aligned}
$$

which yields (3.20). Here the interchange of the order of summation is justified because of the assumption (3.19).

5.3 We first note that

$$
\begin{aligned}
\sum_{l=0}^{\infty}|\tilde{\alpha}_l| &\le \sum_{l=0}^{\infty}\sum_{k=l+1}^{\infty}|\alpha_k| = \sum_{k=1}^{\infty}\sum_{l=0}^{k-1}|\alpha_k| \\
&= \sum_{k=1}^{\infty}k|\alpha_k|\,.
\end{aligned}
$$

This last quantity is finite because of (3.19). Thus $\sum_{l=0}^{\infty} \tilde{\alpha}_l^2 < \infty$ and $\{\tilde{\varepsilon}_j\}$ is well defined in the m.s. sense so that $E(\tilde{\varepsilon}_j) = 0$ and $E(\tilde{\varepsilon}_j \tilde{\varepsilon}_{j+k}) = \sigma^2 \sum_{l=0}^{\infty} \tilde{\alpha}_l \tilde{\alpha}_{l+|k|}$.

5.4 Suppose that $\alpha(L) = \sum_{l=0}^{p} \alpha_l L^l$. Then $\alpha(L) = \alpha(1) - (\alpha(1) - \alpha(L))$, where

$$\begin{aligned} \alpha(1) - \alpha(L) &= \sum_{k=1}^{p} \alpha_k \left(1 - L^k\right) = (1 - L) \sum_{k=1}^{p} \alpha_k \sum_{l=0}^{k-1} L^l \\ &= (1 - L) \sum_{l=0}^{p-1} \sum_{k=l+1}^{p} \alpha_k L^l \,. \end{aligned}$$

Thus we obtain

$$\sum_{l=0}^{p} \alpha_l \, \varepsilon_{j-l} = \alpha(1)\varepsilon_j - (1 - L) \sum_{l=0}^{p-1} \beta_l \, \varepsilon_{j-l} \,,$$

where $\beta_l = \sum_{k=l+1}^{p} \alpha_k$.

5.5 If $\max_{1 \le j \le n} |Z_j| > \delta$, then there exists j such that $|Z_j| > \delta$. Thus $\sum_{j=1}^{n} Z_j^2 I(|Z_j| > \delta)$ $\ge Z_j^2 > \delta^2$. On the other hand, if $\sum_{j=1}^{n} Z_j^2 I(|Z_j| > \delta) > \delta^2$, there must exist j such that $|Z_j| > \delta$. Thus $\max_{1 \le j \le n} |Z_j| > \delta$.

5.6 Using $Y_T(t)$ defined in (3.17) we have

$$\frac{1}{T^2} \sum_{j=1}^{T} y_j^2 - \int_0^1 Y_T^2(t) dt = \sum_{j=1}^{T} \int_{\frac{j-1}{T}}^{\frac{j}{T}} \left[Y_T^2 \left(\frac{j}{T} \right) - Y_T^2(t) \right] dt$$

and

$$\left| Y_T^2 \left(\frac{j}{T} \right) - Y_T^2(t) \right| \le 2 \sup_{0 \le t \le 1} |Y_T(t)| \max_{1 \le j \le T} \frac{|u_j|}{\sqrt{T}} \,.$$

This last quantity converges in probability to 0 because of (3.27) and (3.28) together with $\sup |Y_T(t)| = O_p(1)$. Thus (3.30) follows from Theorem 3.7 with $\sigma = 1$ and the continuous mapping theorem. The weak convergence in (3.31) can be established similarly by following the same lines as in the solution to Problem 4.8.

6.1 Putting $a_j = s_j^2/s_n^2$ we can express $\xi_n(t)$ as $\xi_n(t) = \alpha\xi_n(a_{j-1}) + \beta\xi_n(a_j)$, where $\alpha = (a_j - t)/(a_j - a_{j-1}) \geq 0$ and $\beta = 1 - \alpha$. This means that $\xi_n(t)$ is on the line joining $(a_{j-1},\ \xi_n(a_{j-1}))$ and $(a_j,\ \xi_n(a_j))$.

6.2 When $\{\varepsilon_j\}$ is i.i.d.$(0, \sigma^2)$, we have $s_n^2 = n\sigma^2$ and (3.34) follows from the weak law of large numbers. On the other hand, (3.35) reduces to $E[\varepsilon_1^2 I(|\varepsilon_1| > \sqrt{n}\sigma\delta)] \to 0$ for every δ, which clearly holds because $E(\varepsilon_1^2) < \infty$.

6.3 We first note that

$$\varepsilon_j^2 = \varepsilon_j^2\, I(|\varepsilon_j| \leq \delta s_n) + \varepsilon_j^2 I(|\varepsilon_j| > \delta s_n) \leq \delta^2 s_n^2 + \varepsilon_j^2 I(|\varepsilon_j| > \delta s_n)$$

so that

$$\begin{aligned} \max_{1\leq j\leq n} \frac{E\left(\varepsilon_j^2\right)}{s_n^2} &\leq \delta^2 + \frac{1}{s_n^2} \max_{1\leq j\leq n} E\left[\varepsilon_j^2\, I(|\varepsilon_j| > \delta s_n)\right] \\ &\leq \delta^2 + \frac{1}{s_n^2} \sum_{j=1}^{n} E\left[\varepsilon_j^2\, I(|\varepsilon_j| > \delta s_n)\right], \end{aligned}$$

which implies (3.36) since δ is arbitrary and the Lindeberg condition is imposed.

6.4 The problem is completely the same as in Problem 5.5 by putting $\varepsilon_j = Z_j\, s_n$.

6.5 The relation (3.42) holds since

$$\begin{aligned} E\left[\,|X|\; I(|X| > \delta)\right] &= \int_0^\infty P\left(\,|X|\; I(|X| > \delta) > x\right) dx \\ &= \int_0^\delta P\left(\,|X|\; I(|X| > \delta) > x\right) dx \\ &\qquad + \int_\delta^\infty P\left(\,|X|\; I(|X| > \delta) > x\right) dx \\ &= \delta P\left(\,|X| > \delta\right) + \int_\delta^\infty P\left(\,|X| > x\right) dx\,. \end{aligned}$$

Then the right side of (3.42) is dominated by $cE\left[\eta^2 I(|\eta| > \delta)\right]$ so that (3.41) holds.

6.6 For a given $\gamma > 0$ and sufficiently large $\delta > 0$ we have

$$\begin{aligned} \sup_j E\left(\varepsilon_j^2\right) &= \sup_j \left[E\left[\varepsilon_j^2 I\left(\varepsilon_j^2 > \delta\right)\right] + E\left[\varepsilon_j^2 I\left(\varepsilon_j^2 \leq \delta\right)\right]\right] \\ &\leq \gamma + \delta\,, \end{aligned}$$

which yields the conclusion.

6.7 To show that (3.34) holds with ε_j replaced by $\varepsilon_{jn} = j\eta_j/n$, we use the method of truncation (see, for example, Roussas (1973, p.146)). Define, for any $\delta > 0$,

$$
\begin{aligned}
Y_j &= \begin{cases} \varepsilon_{jn}^2 & (\varepsilon_{jn}^2 \le \delta n)\,, \\[2ex] 0 & (\varepsilon_{jn}^2 > \delta n)\,, \end{cases} \\
Z_j &= \begin{cases} 0 & (\varepsilon_{jn}^2 \le \delta n)\,, \\[2ex] \varepsilon_{jn}^2 & (\varepsilon_{jn}^2 > \delta n)\,, \end{cases}
\end{aligned}
$$

so that $\varepsilon_{jn}^2 = Y_j + Z_j$. Then it holds that

$$
\begin{aligned}
& P\left[\left|\frac{1}{s_n^2}\sum_{j=1}^n \left(\varepsilon_{jn}^2 - E(\varepsilon_{jn}^2)\right)\right| > 3\gamma\right] \\
&\quad \le P\left[\frac{1}{s_n^2}\left\{\left|\sum_{j=1}^n (Y_j - E(Y_j))\right| + \left|\sum_{j=1}^n Z_j\right| + \left|\sum_{j=1}^n (E(Y_j) - E\left(\varepsilon_{jn}^2\right))\right|\right\} > 3\gamma\right] \\
&\quad \le P\left(\frac{1}{s_n^2}\left|\sum_{j=1}^n (Y_j - E(Y_j))\right| > \gamma\right) + P\left(\sum_{j=1}^n Z_j \ne 0\right) \\
&\qquad\qquad + P\left(\frac{1}{s_n^2}\left|\sum_{j=1}^n \left(E(Y_j) - E(\varepsilon_{jn}^2)\right)\right| > \gamma\right).
\end{aligned}
$$

Here the first term is bounded by

$$
\begin{aligned}
\frac{1}{\gamma^2 s_n^4}\sum_{j=1}^n V(Y_j) &\le \frac{1}{\gamma^2 s_n^4}\sum_{j=1}^n E\left[\varepsilon_{jn}^4 I(\varepsilon_{jn}^2 \le \delta n)\right] \\
&\le \frac{\delta n}{\gamma^2 s_n^4}\sum_{j=1}^n E\left[\varepsilon_{jn}^2 I(\varepsilon_{jn}^2 \le \delta n)\right] \\
&\le \frac{\delta n^2 \sigma^2}{\gamma^2 s_n^4} \le \frac{c_1 \delta}{\gamma^2} \qquad \text{for some } c_1 > 0\,.
\end{aligned}
$$

The second term is bounded by

$$
\begin{aligned}
nP(Z_n \ne 0) &= nP\left(\varepsilon_{nn}^2 > \delta n\right) \\
&\le \frac{1}{\delta} E\left[\varepsilon_{nn}^2 I(\varepsilon_{nn}^2 > \delta n)\right] \\
&\le \delta \qquad \text{for } n \text{ sufficiently large.}
\end{aligned}
$$

As for the third term we have

$$\begin{aligned}\frac{1}{s_n^2}\,|\sum_{j=1}^{n}\left(E(Y_j)-E(\varepsilon_{jn}^2)\right)| &= \frac{1}{s_n^2}\sum_{j=1}^{n}E\left[\varepsilon_{jn}^2\,I(\varepsilon_{jn}^2>\delta n)\right]\\ &\le \frac{\delta^2 n}{s_n^2}\le c_2\delta^2\end{aligned}$$

for sufficiently large n and some $c_2>0$. Putting $\delta=\gamma^3$ we now establish (3.34) with ε_j replaced by ε_{jn}. We next consider

$$\begin{aligned}\frac{1}{s_n^2}\sum_{j=1}^{n}E\left[\varepsilon_{jn}^2\,I(|\varepsilon_{jn}|>\delta s_n)\right] &\le \frac{1}{s_n^2}\sum_{j=1}^{n}E\left[\eta_j^2\,I(|\eta_j|>\delta s_n)\right]\\ &= \frac{n}{s_n^2}E\left[\eta_1^2\,I(|\eta_1|>\delta s_n)\right],\end{aligned}$$

which clearly converges to 0 for every $\delta>0$ so that (3.35) holds with ε_j replaced by ε_{jn}.

7.1 Using the BN decomposition $u_i=\alpha\varepsilon_i+\tilde{\varepsilon}_{i-1}-\tilde{\varepsilon}_i$ in (3.20) and substituting this into (3.45) it is easy to establish (3.49).

7.2 The inequality easily follows from the definition of $R_n(t)$ in (3.51) and $0\le(ts_n^2-s_{j-1}^2)/(s_j^2-s_{j-1}^2)\le 1$.

7.3 From the definition of $\{\tilde{\varepsilon}_j\}$ in (3.53) with $\{\varepsilon_j\}$ being a martingale difference it is easy to derive

$$E(\tilde{\varepsilon}_j^2)=\sum_{l=0}^{\infty}\tilde{\alpha}_l^2E(\varepsilon_{j-l}^2)\,.$$

Moreover we have $\sup_j E(\varepsilon_j^2)\le cE(\eta^2)$ so that (3.55) is established.

7.4 We have, for any $\delta>0$,

$$E\left(\tilde{\varepsilon}_j^2I(|\tilde{\varepsilon}_j|>\delta)\right)\le E\left(\tilde{\varepsilon}_j^2\left(\frac{|\tilde{\varepsilon}_j|}{\delta}\right)^{\gamma}I(|\tilde{\varepsilon}_j|>\delta)\right)\le\frac{1}{\delta^{\gamma}}\,E\left(|\tilde{\varepsilon}_j|^{2+\gamma}\right)$$

so that the result follows.

7.5 It can be proved easily that strong uniform integrability of $\{\varepsilon_j\}$ with a bounding variable $\eta(E(|\eta|^{2+\gamma}) < \infty)$ implies $\sup_j E(|\varepsilon_j|^{2+\gamma}) < \infty$. Then it follows from Hölder's inequality that

$$
\text{(S4)} \qquad \begin{aligned} |\tilde{\varepsilon}_j| &\le \sum_{l=0}^{\infty} |\tilde{\alpha}_l|^{\frac{1}{p}} \left(|\tilde{\alpha}_l|^{\frac{1}{q}} |\varepsilon_{j-l}| \right) \qquad \left(\frac{1}{p} + \frac{1}{q} = 1, \;\; q > 1 \right) \\ &\le \left(\sum_{l=0}^{\infty} |\tilde{\alpha}_l| \right)^{\frac{1}{p}} \left(\sum_{l=0}^{\infty} |\tilde{\alpha}_l| \, |\varepsilon_{j-l}|^q \right)^{\frac{1}{q}} \end{aligned}
$$

so that

$$
\sup_j E(|\tilde{\varepsilon}_j|^q) \le \sup_j E(|\varepsilon_j|^q) \left(\sum_{l=0}^{\infty} |\tilde{\alpha}_l| \right)^q .
$$

Putting $q = 2 + \gamma$ we obtain the conclusion.

7.6 Putting $p = q = 2$ in (S4) we obtain

$$
\tilde{\varepsilon}_j^2 \le \sup_j \varepsilon_j^2 \left(\sum_{l=0}^{\infty} |\tilde{\alpha}_l| \right)^2 \equiv |X| \, .
$$

Thus we have

$$
\frac{1}{s_n^2} \sum_{j=1}^{n} E\left[\tilde{\varepsilon}_j^2 \, I\left(\tilde{\varepsilon}_j^2 > s_n^2 \delta \right) \right] \le E\left[|X| \, I\left(|X| > s_n^2 \delta \right) \right] \left(\frac{s_n^2}{n} \right)^{-1} ,
$$

which converges to 0 because $E(|X|) < \infty$ by assumption and s_n^2 is the same order as n.

8.1 Note that $(1-L)y_j^{(d)} = \varepsilon_j/(1-L)^{d-1} = y_j^{(d-1)}$, which yields $y_j^{(d)} = y_{j-1}^{(d)} + y_j^{(d-1)}$. By back substitution this produces $y_j^{(d)} = y_1^{(d-1)} + \cdots + y_j^{(d-1)}$.

8.2 It is easy to establish the first inequality, which is bounded by

$$
\begin{aligned} &\sum_{i=1}^{j} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left| Y_n^{(1)}\left(\frac{i}{n} \right) - Y_n^{(1)}(s) \right| ds + \int_t^{\frac{j}{n}} \left| Y_n^{(1)}(s) \right| ds + \frac{1}{\sqrt{n}} \max_{1 \le j \le n} |\varepsilon_j| \\ &\quad \le \frac{2}{\sqrt{n}} \max_{1 \le j \le n} |\varepsilon_j| + \frac{1}{n} \sup_{0 \le t \le 1} \left| Y_n^{(1)}(t) \right| . \end{aligned}
$$

8.3 It is sufficient to show that the quantities on the right side of (3.63) converge in probability to 0. The first term does because of (3.11) (see Problem 4.6), while $\sup_{0\le t\le 1} |Y_n^{(1)}(t)| = O_p(1)$ because $\mathcal{L}(\sup_{0\le t\le 1} |Y_n^{(1)}(t)|) \to \mathcal{L}(\sigma \sup_{0\le t\le 1} |w(t)|)$. Thus the second term also converges in probability to 0.

8.4 It is obvious that the first inequality holds. Since

$$y_j^{(k)} = \frac{\varepsilon_j}{(1-L)^k} = \Sigma \cdots \Sigma\, \varepsilon_i\,, \qquad (k \quad \Sigma\text{'s})\,,$$

it holds that $|y_j^{(k)}| \le n^k \max_{1\le j\le n} |\varepsilon_j|$, which establishes the second inequality.

8.5 Using the BN decomposition we obtain

$$\text{(S5)} \qquad \begin{aligned} y_j^{(d)} &= \frac{u_j}{(1-L)^d} = \frac{1}{(1-L)^d}\,[\alpha\varepsilon_j - (1-L)\tilde{\varepsilon}_j] \\ &= \alpha\,\frac{\varepsilon_j}{(1-L)^d} - \frac{\tilde{\varepsilon}_j}{(1-L)^{d-1}} = \alpha x_j^{(d)} - z_j^{(d-1)}\,, \end{aligned}$$

where $(1-L)^d\, x_j^{(d)} = \varepsilon_j$, $(1-L)^{d-1} z_j^{(d-1)} = \tilde{\varepsilon}_j$. Therefore we have

$$\begin{aligned} Y_n^{(d)}(t) &= \frac{1}{n^{d-\frac{1}{2}}}\, y_{[nt]}^{(d)} + (nt-[nt])\,\frac{1}{n^{d-\frac{1}{2}}}\, y_{[nt]+1}^{(d-1)} \\ &= \alpha \left[\frac{1}{n^{d-\frac{1}{2}}}\, x_{[nt]}^{(d)} + (nt-[nt])\frac{1}{n^{d-\frac{1}{2}}}\, x_{[nt]+1}^{(d-1)}\right] + R_n(t)\,, \end{aligned}$$

where

$$\begin{aligned} |R_n(t)| &= \frac{1}{n^{d-\frac{1}{2}}} \left| z_{[nt]}^{(d-1)} + (nt-[nt])\, z_{[nt]+1}^{(d-2)} \right| \\ &\le \frac{1}{\sqrt{n}} \max_{0\le j\le n} |\, \tilde{\varepsilon}_j \,| + \frac{1}{n\sqrt{n}} \max_{0\le j\le n} |\, \tilde{\varepsilon}_j \,|\,. \end{aligned}$$

Since $\sup_{0\le t\le 1} |R_n(t)|$ converges in probability to 0 (see (3.27)), Theorem 3.10 follows from the result for the case $u_j = \varepsilon_j$ and the continuous mapping theorem.

8.6 We have only to show that the right side of (3.72) converges in probability to 0. Since $\mathcal{L}\left(\sup_{0\le t\le 1} |Y_T^{(d)}(t)|\right) \to \mathcal{L}\left(|\alpha|\sigma \sup_{0\le t\le 1} |F_{d-1}(t)|\right)$ so that $\sup_{0\le t\le 1} |Y_T^{(d)}(t)| = O_p(1)$ and

$$\frac{1}{T^{d-\frac{1}{2}}} \max_{1\le j\le T} \left|y_j^{(d-1)}\right| \le \frac{1}{\sqrt{T}} \max_{1\le j\le T} |u_j|\,,$$

we obtain the conclusion noting that $\max_{1\le j\le T} |u_j|/\sqrt{T}$ converges in probability to 0 because of strict and second-order stationarity of $\{u_j\}$.

9.1 Abel's transformation corresponds to the partial integration formula and can be proved easily. Putting $a_i = \rho_n^{j-i}$ and $b_i = S_i$, we obtain

$$\begin{aligned} a_{j+1}b_j - a_1 b_0 &= \rho_n^{-1} S_j \,, \\ (a_{i+1} - a_i)\, b_i &= (1-\rho_n)\, \rho_n^{j-i-1} S_i \,, \end{aligned}$$

which establishes (3.78).

9.2 Consider $|h_t(x;\gamma) - h_t(y;\gamma)|$ for x, $y \in C$, which is bounded by $(2 + e^{|\beta|})\rho(x, y)$ so that h is a continuous mapping defined on C.

9.3 The partial integration formula yields

$$\int_0^t e^{\beta s} dw(s) = e^{\beta t} w(t) - \beta \int_0^t e^{\beta s} w(s) ds \,,$$

which leads us to the conclusion.

9.4 We need to prove that the right side of (3.84) converges in probability to 0 uniformly in j as $n \to \infty$. Consider

$$A_{jn} \le \sup_{0\le t\le 1} \left| \exp\left\{ [nt] \log\left(1 - \frac{\beta}{n}\right)\right\} - e^{-\beta t} \right| ,$$

where $\log\left(1 - \dfrac{\beta}{n}\right) = -\dfrac{\beta}{n} + O(n^{-2})$. Then it holds that

$$A_{jn} \le c\,|\exp(O(n^{-1})) - 1| \longrightarrow 0$$

with c being a positive constant. We can show similarly that $C_{jn} \to 0$ in probability, while it is almost obvious that $B_{jn} \to 0$ and $D_{jn} \to 0$ in probability.

9.5 We have

$$|V_T - h(X_T)| = \left| \frac{1}{T} \sum_{j=1}^{T} X_T^2\left(\frac{j}{T}\right) - \int_0^1 X_T^2(t) dt \right|$$

$$
\begin{aligned}
&\le 2 \sup_{0\le t\le 1} |X_T(t)| \frac{1}{\sqrt{T}} \max_{1\le j\le T} |y_j - y_{j-1}| \\
&\le 2 \sup_{0\le t\le 1} |X_T(t)| \left[\frac{|\beta|}{T} \sup_{0\le t\le 1} |X_T(t)| + \frac{1}{\sqrt{T}} \max_{1\le j\le T} |\varepsilon_j| \right] ,
\end{aligned}
$$

which converges in probability to 0. Then $\mathcal{L}(V_T/\sigma^2) \to \mathcal{L}(h(X))$ since $\mathcal{L}(X_T/\sigma) \to \mathcal{L}(X)$ and $\mathcal{L}(h(X_T/\sigma)) \to \mathcal{L}(h(X))$.

9.6 The stochastic process $\{X_n(t)\}$ in the theorem can be rewritten as in (3.86) with $X(0)$ replaced by $\alpha X(0)$, where $Y_n(t)$ is now defined, using the BN decomposition, as

$$
\begin{aligned}
Y_n(t) &= \frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} u_j + (nt - [nt]) \frac{1}{\sqrt{n}} u_{[nt]+1} \\
&= \alpha Z_n(t) + R_n(t) .
\end{aligned}
$$

Here $Z_n(t)$ is given by the right side of (3.82), while $R_n(t)$ is the remainder term defined by (3.25). We also have

$$
\begin{aligned}
y_j &= \alpha x_j + \sum_{i=1}^{j} \rho_n^{j-i} (\tilde{\varepsilon}_{i-1} - \tilde{\varepsilon}_i) , \\
x_j &= \left(1 - \frac{\beta}{n}\right) x_{j-1} + \varepsilon_j , \qquad x_0 = \sqrt{n}\, \sigma X(0) .
\end{aligned}
$$

Then $X_n(t) = \alpha U_n(t) + M_n(t)$, where

$$
\begin{aligned}
U_n(t) &= \rho_n^{j-1} \sigma X(0) + \rho_n^{-1} Z_n\left(\frac{j-1}{n}\right) - \frac{\beta}{n} \sum_{i=1}^{j-1} \rho_n^{j-i-2} Z_n\left(\frac{i}{n}\right) \\
&\quad + n \left(t - \frac{j-1}{n}\right) \frac{x_j - x_{j-1}}{\sqrt{n}} , \\
M_n(t) &= \rho_n^{-1} R_n\left(\frac{j-1}{n}\right) - \frac{\beta}{n} \sum_{i=1}^{j-1} \rho_n^{j-i-2} R_n\left(\frac{i}{n}\right) \\
&\quad + n \left(t - \frac{j-1}{n}\right) \frac{1}{\sqrt{n}} \left(\sum_{i=1}^{j} \rho_n^{j-i} (\tilde{\varepsilon}_{i-1} - \tilde{\varepsilon}_i) - \sum_{i=1}^{j-1} \rho_n^{j-i-1} (\tilde{\varepsilon}_{i-1} - \tilde{\varepsilon}_i) \right) .
\end{aligned}
$$

Using the fact that $\sup_{0\le t\le 1} |R_n(t)| \to 0$ in probability, we can show that $\sup_{0\le t\le 1} |M_n(t)| \to 0$ in probability. Thus $\mathcal{L}(X_n/\sigma) \to \mathcal{L}(\alpha X)$ by the continuous mapping theorem since $\mathcal{L}(U_n/\sigma) \to \mathcal{L}(X)$.

10.1 Define

$$h(x) = \mathrm{tr}\left(\int_0^1 x(t)x'(t)dt\right), \qquad x \in C^q .$$

Then we note that

$$\frac{1}{T^2}\sum_{j=1}^{T} y_j' H' H y_j = \mathrm{tr}\left(H'H\frac{1}{T^2}\sum_{j=1}^{T} y_j y_j'\right),$$

which converges in distribution to $h(H\Sigma^{\frac{1}{2}}\underset{\sim}{w})$. This establishes (3.94).

10.2 We first note that

$$\begin{aligned}\sum_{l=0}^{\infty}\|\tilde{A}_l\| &\le \sum_{l=0}^{\infty}\sum_{k=l+1}^{\infty}\|A_k\| = \sum_{k=1}^{\infty}\sum_{l=0}^{k-1}\|A_k\| \\ &= \sum_{k=1}^{\infty} k\|A_k\| < \infty .\end{aligned}$$

Thus $\sum_{l=0}^{\infty} A_l A_l'$ converges and $\{\tilde{\varepsilon}_j\}$ is well defined in the m.s. sense so that $E(\tilde{\varepsilon}_j) = 0$ and

$$E(\tilde{\varepsilon}_j\,\tilde{\varepsilon}_{j+k}') = \begin{cases} \displaystyle\sum_{l=0}^{\infty} A_l A_{l+k}' & (k \ge 0), \\ \displaystyle\sum_{l=0}^{\infty} A_{l-k} A_l' & (k < 0). \end{cases}$$

10.3 The inequality follows from the triangle inequality and the Cauchy-Schwarz inequality.

10.4 Using the relation

$$aa' - bb' = (a-b)(a-b)' + b(a-b)' + (a-b)b' ,$$

it is easy to obtain that, for x fixed,

$$\|h(x) - h(y)\| \le \left[\rho_q^2(x,y) + 2\rho_q(x,y)\sup_{0\le t\le 1}\|x(t)\|\right] \times q$$

so that h is a continuous mapping defined on C^q.

10.5 The first inequality is obvious, while the second comes from the fact that

$$\left| Y_{kT}\left(\frac{j}{T}\right) - Y_{kT}(t)\right| \le \|A^{-1}\| \frac{1}{\sqrt{T}} \max_{1\le j\le T} \|u_j\| \,.$$

The right side above converges in probability to 0 if $\sum_{j=1}^{T} u_j' u_j I(u_j' u_j > T\delta)/T$ converges in probability to 0 for any $\delta > 0$, which follows from second-order stationarity of $\{u_j\}$ and the Markov inequality.

10.6 Define $x_j^{(d)} = \varepsilon_j/(1-L)^d$ with $x_{-(d-1)}^{(d)} = x_{-(d-2)}^{(d)} = \cdots = x_0^{(d)} = 0$ and put, for $d \ge 2$,

$$\begin{aligned} X_n^{(d)}(t) &= \frac{1}{n^{d-\frac{1}{2}}} x_{[nt]}^{(d)} + (nt - [nt]) \frac{1}{n^{d-\frac{1}{2}}} x_{[nt]+1}^{(d-1)} \\ &= \frac{1}{n} \sum_{j=1}^{[nt]} X_n^{(d-1)}\left(\frac{j}{n}\right) + (nt-[nt]) \frac{1}{n^{d-\frac{1}{2}}} x_{[nt]+1}^{(d-1)} \,, \end{aligned}$$

where

$$X_n^{(1)}(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} \varepsilon_j + (nt-[nt]) \frac{1}{\sqrt{n}} \varepsilon_{[nt]+1} \,.$$

Using the BN decomposition we have $y_j^{(d)} = u_j/(1-L)^d = Ax_j^{(d)} - z_j^{(d-1)}$ with $z_j^{(d-1)} = \tilde{\varepsilon}_j/(1-L)^{d-1}$ so that $Y_n^{(d)}(t) = AX_n^{(d)}(t) + R_n(t)$, where

$$\begin{aligned} |R_{in}(t)| &\le \frac{1}{n^{d-\frac{1}{2}}} \left\| z_{[nt]}^{(d-1)} + (nt-[nt])\, z_{[nt]+1}^{(d-2)} \right\| \\ &\le \frac{1}{\sqrt{n}} \max_{0\le j\le n} \|\tilde{\varepsilon}_j\| + \frac{1}{n\sqrt{n}} \max_{0\le j\le n} \|\tilde{\varepsilon}_j\| \,. \end{aligned}$$

It is seen that $\rho_q(Y_n^{(d)}\,,\ AX_n^{(d)}) \to 0$ in probability. Define now

$$G_{dn}(t) = \int_0^t X_n^{(d)}(s)ds \,,$$

where $\mathcal{L}(G_{1n}) \to \mathcal{L}(F_1)$. Since it can be shown that $\rho(X_n^{(2)}\,,\ G_{1n}) \to 0$ in probability, it holds that $\mathcal{L}(Y_n^{(2)}) \to \mathcal{L}(AF_1)$.

10.7 Suppose that the theorem holds for $d = k-1$ (≥ 3). Using the notations in the solution to Problem 10.6, we have $Y_n^{(k)}(t) = AX_n^{(k)}(t) + R_n(t)$ with $\rho_q(Y_n^{(k)}\,,\ AX_n^{(k)}) \to 0$ in probability and

$$\|X_n^{(k)}(t) - G_{k-1,n}(t)\| \le \frac{2}{\sqrt{n}} \max_{1\le j\le n} \|\varepsilon_j\| + \frac{1}{n} \sup_{0\le t\le 1} \|X_n^{(k-1)}(t)\| \,,$$

which converges in probability to 0. Since $\mathcal{L}(X_n^{(k-1)}) \to \mathcal{L}(F_{k-2})$ and $\mathcal{L}(G_{k-1,n}) \to \mathcal{L}(F_{k-1})$ by assumption, we can conclude that $\mathcal{L}(X_n^{(k)}) \to \mathcal{L}(F_{k-1})$ and thus $\mathcal{L}(Y_n^{(k)}) \to \mathcal{L}(AF_{k-1})$.

10.8 The first inequality is obvious. The second inequality can be obtained by using the relation $aa' - bb' = (a-b)(a-b)' + b(a-b)' + (a-b)b'$.

10.9 Note that, for $(j-1)/T \le t \le j/T$,

$$Y_T^{(d)}(t) = \frac{1}{T^{d-\frac{1}{2}}}\, y_{j-1}^{(d)} + T\left(t - \frac{j-1}{T}\right)\frac{1}{T^{d-\frac{1}{2}}}\, y_j^{(d-1)},$$

so that $dY_T^{(d)}(t)/dt = y_j^{(d-1)}/T^{d-\frac{3}{2}} = (y_j^{(d)} - y_{j-1}^{(d)})/T^{d-\frac{3}{2}}$. Thus the left side of (3.110) is equal to

$$\begin{aligned}
&\sum_{j=1}^{T}\int_{\frac{j-1}{T}}^{\frac{j}{T}}\left[\frac{y_{j-1}^{(d)}}{T^{d-\frac{1}{2}}} + T\left(t - \frac{j-1}{T}\right)\frac{y_j^{(d-1)}}{T^{d-\frac{1}{2}}}\right] dt\, \frac{(y_j^{(d)} - y_{j-1}^{(d)})'}{T^{d-\frac{3}{2}}} \\
&= \frac{1}{T^{2d-1}}\sum_{j=1}^{T} y_{j-1}^{(d)}\left(y_j^{(d)} - y_{j-1}^{(d)}\right)' + \frac{1}{2T^{2d-1}}\sum_{j=1}^{T} y_j^{(d-1)}\left(y_j^{(d)} - y_{j-1}^{(d)}\right)' \\
&= U_T^{(d)} + \frac{1}{2T^{2d-1}}\sum_{j=1}^{T} y_j^{(d-1)}\left(y_j^{(d-1)}\right)'.
\end{aligned}$$

10.10 It holds that

$$\begin{aligned}
&\sum_{j=1}^{T}\left[y_{j-1}^{(d)}\left(y_j^{(d)} - y_{j-1}^{(d)}\right)' + \left(y_j^{(d)} - y_{j-1}^{(d)}\right)\left(y_{j-1}^{(d)}\right)'\right] \\
&= -\left[\sum_{j=1}^{T}\left(y_j^{(d)} - y_{j-1}^{(d)}\right)\left(y_j^{(d)} - y_{j-1}^{(d)}\right)' - \sum_{j=1}^{T} y_j^{(d)}\left(y_j^{(d)}\right)' + \sum_{j=1}^{T} y_{j-1}^{(d)}\left(y_{j-1}^{(d)}\right)'\right]
\end{aligned}$$

so that

$$\begin{aligned}
U_T^{(d)} + \left(U_T^{(d)}\right)' &= \frac{1}{T^{2d-1}}\, y_T^{(d)}\left(y_T^{(d)}\right)' - \frac{1}{T^{2d-1}}\sum_{j=1}^{T} y_j^{(d-1)}\left(y_j^{(d-1)}\right)' \\
&= Y_T^{(d)}(1)\left(Y_T^{(d)}(1)\right)' + O_p\left(\frac{1}{T}\right).
\end{aligned}$$

We now have (3.111) because of (3.108).

11.1 Noting that $dX_T(t)/dt = \sqrt{T}(x_j - x_{j-1})$ for $(j-1)/T \le t \le j/T$, we have

$$\begin{aligned}
&\int_0^1 X_T(t)dX_T(t) \\
&= \sum_{j=1}^{T} \int_{\frac{j-1}{T}}^{\frac{j}{T}} \left[\frac{x_{j-1}}{\sqrt{T}} + T\left(t - \frac{j-1}{T}\right)\frac{x_j - x_{j-1}}{\sqrt{T}}\right] dt \, \sqrt{T}(x_j - x_{j-1}) \\
&= \frac{1}{T}\sum_{j=1}^{T} x_{j-1}(x_j - x_{j-1}) + \frac{1}{2T}\sum_{j=1}^{T}(x_j - x_{j-1})^2 ,
\end{aligned}$$

which yields the right side of (3.113).

11.2 Noting that $\varepsilon_j = x_j - x_{j-1} + \beta x_{j-1}/T$ we have

$$\text{(S6)} \qquad \frac{1}{T}\sum_{j=1}^{T} x_{j-1}\varepsilon_j = \frac{1}{T}\sum_{j=1}^{T} x_{j-1}(x_j - x_{j-1}) + \frac{\beta}{T^2}\sum_{j=1}^{T} x_{j-1}^2 ,$$

where

$$\mathcal{L}\left(\frac{1}{T}\sum_{j=1}^{T} x_{j-1}(x_j - x_{j-1}) , \ \frac{\beta}{T^2}\sum_{j=1}^{T} x_{j-1}^2\right) \longrightarrow \mathcal{L}\left(\int_0^1 X(t)dX(t) , \ \beta\int_0^1 X^2(t)dt\right) .$$

It follows from the continuous mapping theorem that

$$\begin{aligned}
\mathcal{L}\left(\frac{1}{T}\sum_{j=1}^{T} x_{j-1}\varepsilon_j\right) \quad &\longrightarrow \quad \mathcal{L}\left(\int_0^1 X(t)dX(t) + \beta\int_0^1 X^2(t)dt\right) \\
&= \quad \mathcal{L}\left(\int_0^1 X(t)dw(t)\right) .
\end{aligned}$$

11.3 Note that

$$\begin{aligned}
\frac{1}{T}\sum_{j=1}^{T} x_{j-1}(x_j - x_{j-1}) \quad &= \quad \frac{1}{2T}(x_T^2 - x_0^2) - \frac{1}{2T}\sum_{j=1}^{T}\left(-\frac{\beta}{T}x_{j-1} + u_j\right)^2 \\
&= \quad \frac{1}{2T}(x_T^2 - x_0^2) - \frac{1}{2T}\sum_{j=1}^{T} u_j^2 + R_T ,
\end{aligned}$$

where

$$R_T = -\frac{\beta^2}{2T^3}\sum_{j=1}^{T} x_{j-1}^2 + \frac{\beta}{T^2}\sum_{j=1}^{T} x_{j-1}u_j .$$

It follows from the results in Section 9 that $R_T \to 0$ in probability and

$$\mathcal{L}\left(\frac{1}{T}\, x_T^2\,,\ \frac{1}{T}\, x_0^2\right) \longrightarrow \mathcal{L}\left(\alpha^2 X^2(1)\,,\ \alpha^2 X^2(0)\right)\,,$$

$$\frac{1}{T}\sum_{j=1}^{T} u_j^2 \longrightarrow \sum_{l=0}^{\infty} \alpha_l^2 \qquad \text{in probability.}$$

Thus Theorem 3.15 follows.

11.4 Since (S6) holds with ε_j replaced by u_j and

$$\mathcal{L}\left(\frac{1}{T}\sum_{j=1}^{T} x_{j-1}(x_j - x_{j-1})\,,\ \frac{\beta}{T^2}\sum_{j=1}^{T} x_{j-1}^2\right)$$
$$\longrightarrow \mathcal{L}\left(\alpha^2\int_0^1 X(t)dX(t) + \frac{1}{2}\left(\alpha^2 - \sum_{l=0}^{\infty}\alpha_l^2\right),\ \alpha^2\beta\int_0^1 X^2(t)dt\right)\,,$$

(3.117) follows from the continuous mapping theorem.

11.5 Since it holds that

$$\frac{1}{T}\sum_{j=1}^{T} y_j' H \varepsilon_j = \text{tr}\,\left(H\frac{1}{T}\sum_{j=1}^{T}\varepsilon_j y_{j-1}'\right) + \text{tr}\,\left(H\frac{1}{T}\sum_{j=1}^{T}\varepsilon_j\varepsilon_j'\right)\,,$$

(3.120) follows from (3.119), the weak law of large numbers and the continuous mapping theorem.

11.6 The first equality is obvious and the remainder term in the second equality is

$$\begin{aligned}
& \frac{1}{T}\left[A\sum_{j=1}^{T} z_{j-1}(\tilde{\varepsilon}_{j-1} - \tilde{\varepsilon}_j)' - A\sum_{j=1}^{T}\varepsilon_j\tilde{\varepsilon}_j' + \tilde{\varepsilon}_0\sum_{j=1}^{T} u_j'\right] \\
= \ & \frac{1}{T}\left[A\sum_{j=1}^{T} z_{j-1}\tilde{\varepsilon}_{j-1}' - A\sum_{j=1}^{T}(z_j - \varepsilon_j)\tilde{\varepsilon}_j' - A\sum_{j=1}^{T}\varepsilon_j\tilde{\varepsilon}_j' + \tilde{\varepsilon}_0\sum_{j=1}^{T} u_j'\right] \\
= \ & \frac{1}{T}\left[A(z_0\tilde{\varepsilon}_0' - z_T\tilde{\varepsilon}_T') + \tilde{\varepsilon}_0\sum_{j=1}^{T}\tilde{u}_j'\right]\,,
\end{aligned}$$

which is evidently $o_p(1)$.

11.7 Define $v_j = (\varepsilon_j', \tilde{\varepsilon}_j')'$ and $w_j = (\tilde{\varepsilon}_{j-1}', u_j')'$. Then $\{v_j\}$ and $\{w_j\}$ are strictly stationary with zero means and finite second moments. It now follows from Theorem 2 of Hannan (1970, p.203) that

$$\begin{aligned}
\frac{1}{T}\sum_{j=1}^{T}\varepsilon_j\tilde{\varepsilon}_j' &\longrightarrow E(\varepsilon_j\tilde{\varepsilon}_j') = E\left[\varepsilon_j\left(\sum_{k=0}^{\infty}\tilde{A}_k\,\varepsilon_{j-k}\right)'\right] \qquad \text{(a.s.)}\\
&= \tilde{A}_0' = (A - A_0)',\\
\frac{1}{T}\sum_{j=1}^{T}\tilde{\varepsilon}_{j-1}\,u_j' &\longrightarrow E(\tilde{\varepsilon}_{j-1}u_j') = E\left[\sum_{l=0}^{\infty}\tilde{A}_l\,\varepsilon_{j-1-l}\left(\sum_{m=0}^{\infty}A_m\varepsilon_{j-m}\right)'\right] \qquad \text{(a.s.)}\\
&= \sum_{l=0}^{\infty}\tilde{A}_l\,A_{l+1}' = \sum_{l=0}^{\infty}\left(\sum_{k=l+1}^{\infty}A_k\right)A_{l+1}'\,.
\end{aligned}$$

11.8 We have only to show that

$$\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\alpha_l\,\alpha_{k+l} = \frac{1}{2}\left(\alpha^2 - \sum_{l=0}^{\infty}\alpha_l^2\right).$$

Since the right side is equal to

$$\sum_{l<k}\alpha_l\,\alpha_k = \sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\alpha_l\,\alpha_{k+l}\,,$$

we have the conclusion.

Chapter 4.

1.1 It is easy to obtain

$$\frac{l_n(\alpha)}{l_n(\beta)} = \exp\left[\frac{\beta-\alpha}{n}\sum_{j=1}^{n}y_{j-1}(y_j - y_{j-1}) - \frac{\alpha^2-\beta^2}{2n^2}\sum_{j=1}^{n}y_{j-1}^2\right],$$

where it holds by the result of Section 3.11 and the continuous mapping theorem that

$$\mathcal{L}\left(\frac{1}{n}\sum_{j=1}^{n}y_{j-1}(y_j - y_{j-1}),\ \frac{1}{n^2}\sum_{j=1}^{n}y_{j-1}^2\right) \longrightarrow \mathcal{L}\left(\int_0^1 y(t)dy(t),\ \int_0^1 y^2(t)dt\right).$$

Thus we establish (4.7) using again the continuous mapping theorem.

1.2 The expressions up to the second last line are a consequence of Theorem 4.1 and the Ito calculus $d\left(Y^2(t)\right) = 2Y(t)dY(t) + dt$. Since

$$Y(1) = \kappa e^{-\beta} + e^{-\beta}\int_0^1 e^{\beta s}dw(s) \sim \mathrm{N}(\mu, \sigma^2)\,,$$

where $\mu = \kappa e^{-\beta}$ and $\sigma^2 = (1 - e^{-2\beta})/(2\beta)$, we have

$$E\left(e^{\theta S_1}\right) = \left[1 - (\beta - \alpha)\sigma^2\right]^{-\frac{1}{2}} \exp\left[\frac{\alpha - \beta}{2}\left(\kappa^2 + 1 - \frac{\mu^2}{1 - (\beta - \alpha)\sigma^2}\right)\right]\,,$$

which yields the last expression in (4.8).

1.3 Putting $X = Y(1)$ and $Y = \displaystyle\int_0^1 Y(t)dt$ we have that $(X, Y)' \sim \mathrm{N}(\mu, \Sigma)$, where

$$\mu = \begin{pmatrix} \kappa e^{-\beta} \\ (1 - e^{-\beta})\dfrac{\kappa}{\beta} \end{pmatrix},$$

$$\Sigma = \begin{pmatrix} \dfrac{1 - e^{-2\beta}}{2\beta} & \dfrac{1}{\beta^2}\left(\dfrac{1}{2} - e^{-\beta} + \dfrac{e^{-2\beta}}{2}\right) \\ \dfrac{1}{\beta^2}\left(\dfrac{1}{2} - e^{-\beta} + \dfrac{e^{-2\beta}}{2}\right) & \dfrac{1}{\beta^3}\left(\beta - \dfrac{3}{2} + 2e^{-\beta} - \dfrac{e^{-2\beta}}{2}\right) \end{pmatrix}.$$

Therefore we obtain

$$E\left[\exp\left\{\frac{\beta - \alpha}{2}X^2 - \theta Y^2\right\}\right] = |\, I_2 - \Sigma\Lambda|^{-\frac{1}{2}} \exp\left[\frac{1}{2}\mu'\Sigma^{-1}\{(I_2 - \Sigma\Lambda)^{-1} - I_2\}\mu\right],$$

where $\Lambda = \mathrm{diag}(\beta - \alpha,\ -2\theta)$. We can arrive at the last expression in (4.11) after some manipulations.

1.4 Using Theorem 4.1 we have

$$E\left[\exp\left\{\theta\int_0^1 (w(t) - tw(1))^2 dt\right\}\right]$$
$$= E\left[\exp\left\{\left(\frac{\beta}{2} + \frac{\theta}{3}\right)Y^2(1) - 2\theta Y(1)\int_0^1 tY(t)dt - \frac{\beta}{2}\right\}\right]\,,$$

where $\beta = \sqrt{-2\theta}$. Putting

$$X = Y(1) = e^{-\beta}\int_0^1 e^{\beta s}dw(s)\,,$$

$$Y = \int_0^1 tY(t)dt = \int_0^1 \left\{\frac{s}{\beta} + \frac{1}{\beta^2} - \left(\frac{1}{\beta} + \frac{1}{\beta^2}\right)e^{-\beta(1-s)}\right\}dw(s)\,,$$

we obtain

$$V(X) = \frac{1 - e^{-2\beta}}{2\beta},$$

$$\mathrm{Cov}(X, Y) = \frac{1}{2\beta^2} - \frac{1}{2\beta^3} + \left(\frac{1}{2\beta^2} + \frac{1}{2\beta^3}\right) e^{-2\beta},$$

$$V(Y) = \frac{1}{3\beta^2} - \frac{1}{2\beta^3} + \frac{1}{2\beta^5} - \left(\frac{1}{2\beta^3} + \frac{1}{\beta^4} + \frac{1}{2\beta^5}\right) e^{-2\beta},$$

from which we can arrive at the result after some algebra.

1.5 We obtain

$$V(Y(0)) = \frac{1}{2\alpha}, \qquad V(Y(1)) = \frac{e^{-2\beta}}{2\alpha} + \frac{1 - e^{-2\beta}}{2\beta},$$

$$V\left(\int_0^1 Y(t)dt\right) = \frac{1}{\beta^2}\left[1 + \frac{1}{2\alpha} - \frac{3}{2\beta} + \left(\frac{2}{\beta} - \frac{1}{\alpha}\right) e^{-\beta} + \left(\frac{1}{2\alpha} - \frac{1}{2\beta}\right) e^{-2\beta}\right],$$

$$\mathrm{Cov}(Y(0),\, Y(1)) = \frac{e^{-\beta}}{2\alpha}, \quad \mathrm{Cov}\left(Y(0),\, \int_0^1 Y(t)dt\right) = \frac{1}{2\alpha\beta}(1 - e^{-\beta}),$$

$$\mathrm{Cov}\left(Y(1),\, \int_0^1 Y(t)dt\right) = \frac{1}{\beta^2}\left[\frac{1}{2} + \left(\frac{\beta}{2\alpha} - 1\right) e^{-\beta} + \left(\frac{1}{2} - \frac{\beta}{2\alpha}\right) e^{-2\beta}\right].$$

Noting that the above three random variables are normally distributed with means 0, some manipulations yield (4.14) and (4.15).

1.6 We can proceed in the same way as in the solution to Problem 1.2. A different definition of β gives a different final expression.

1.7 Since $E[\exp(\theta_1 U + \theta_2 S_1)] = E[\exp\{\theta(xS_1 - U)\}]$ with $\theta = -\theta_1$ and $x = -\theta_2/\theta_1$, the joint m.g.f. of U and S_1 is derived from (4.17) replacing θ, x and β by $-\theta_1$, $-\theta_2/\theta_1$ and $\sqrt{\alpha^2 - 2\theta_2}$, respectively.

1.8 Replacing κ in (4.17) by $X(0)$ and noting that $E\left[\exp\left\{\theta X^2(0)\right\}\right] = (1 - \theta/\alpha)^{-\frac{1}{2}}$, we obtain, from (4.17),

$$E\left[\exp\left\{\theta(xS_1 - U)\right\}\right] = E\left[E\left[\exp\left\{\theta(xS_1 - U)\right\} \mid X(0)\right]\right]$$

$$= \exp\left(\frac{\alpha+\theta}{2}\right)\left[\left(1-\frac{\theta\left(\alpha+\dfrac{\theta}{2}+x\right)}{\alpha g(\theta)}\,\frac{\sinh\beta}{\beta}\right)g(\theta)\right]^{-\frac{1}{2}}$$

$$= \exp\left(\frac{\alpha+\theta}{2}\right)\left[g(\theta)-\frac{\theta\left(\alpha+\dfrac{\theta}{2}+x\right)}{\alpha}\,\frac{\sinh\beta}{\beta}\right]^{-\frac{1}{2}},$$

where $g(\theta)=\cosh\beta+(\alpha+\theta)\sinh\beta/\beta$. This gives us (4.19).

1.9 Noting that

$$T(\hat{\rho}-1)=\frac{\dfrac{1}{T}\displaystyle\sum_{j=1}^{T}y_{j-1}(y_j-y_{j-1})}{\dfrac{1}{T^2}\displaystyle\sum_{j=1}^{T}y_{j-1}^2},$$

$$\mathcal{L}\left(\frac{1}{T}\sum_{j=1}^{T}y_{j-1}(y_j-y_{j-1})\,,\ \frac{1}{T^2}\sum_{j=1}^{T}y_{j-1}^2\right)\longrightarrow\mathcal{L}\left(\int_0^1 X(t)dX(t)\,,\ \int_0^1 X^2(t)dt\right),$$

we can establish (4.20) by the continuous mapping theorem.

1.10 By the conditional argument leading to (4.24) we have

$$E\left[\exp\{i\theta(xS_1-V)\}\right]=E\left[\exp\left\{i\left(\theta x+\frac{i\theta^2}{2}\right)S_1\right\}\right]$$

and thus, replacing θ by $i\left(\theta x+(i\theta^2)/2\right)$ in (4.14), we obtain (4.25). Note that $\cosh\sqrt{-\theta}=\cos\sqrt{\theta}$ and $\sinh\sqrt{-\theta}/\sqrt{-\theta}=\sin\sqrt{\theta}/\sqrt{\theta}$.

1.11 Noting that

$$T(\hat{\beta}-\beta)=\frac{\dfrac{1}{T}\displaystyle\sum_{j=1}^{T}y_{1j}\varepsilon_{2j}}{\dfrac{1}{T^2}\displaystyle\sum_{j=1}^{T}y_{1j}^2},$$

$$\mathcal{L}\left(\frac{1}{T}\sum_{j=1}^{T} y_{1j}\varepsilon_{2j}\,,\ \frac{1}{T^2}\sum_{j=1}^{T} y_{1j}^2\right) \longrightarrow \mathcal{L}\left(\int_0^1 X_1(t)dw_2(t)\,,\ \int_0^1 X_1^2(t)dt\right)\,,$$

we can establish (4.26) by the continuous mapping theorem.

2.1 Since $dY_g(t)/dt = \beta Y_g(t) + F_{g-1}(t)$ and $d^{g-1}F_{g-1}(t)/dt^{g-1} = w(t)$, (4.33) follows by differentiation.

2.2 Noting that $(1-L)^g(1-(1+(\beta/T))L)y_j = (1-(1+(\beta/T)L)z_j = \varepsilon_j$ we obtain

$$\left.\frac{l_T(0)}{l_T(\alpha)}\right|_\beta = \exp\left[\frac{-\alpha}{T}\sum_{j=1}^{T} z_{j-1}(z_j - z_{j-1}) + \frac{\alpha^2}{2T^2}\sum_{j=1}^{T} z_{j-1}^2\right]\,,$$

where $z_j = (1+(\beta/T))z_{j-1} + \varepsilon_j$. Therefore (4.39) holds by the continuous mapping theorem. Since $z_j = (1-L)^g y_j$, $\{Z(t)\}$ must satisfy $Z(t) = d^g Y_g(t)/dt^g$.

2.3 The expressions up to the third equality are a consequence of Theorem 4.2. Since $dY_1(t) = (\beta Y_1(t) + w(t))dt$, we have

$$\int_0^1 \frac{dY_1(t)}{dt} d\left(\frac{dY_1(t)}{dt}\right) = \frac{1}{2}\left[(\beta Y_1(1) + w(1))^2 - 1\right]\,,$$

$$\begin{aligned}
\int_0^1 \left(\frac{dY_1(t)}{dt}\right)^2 dt &= \int_0^1 (\beta Y_1(t) + w(t))^2\, dt \\
&= \beta^2 \int_0^1 Y_1^2(t)dt + 2\beta \int_0^1 Y_1(t)\left(dY_1(t) - \beta Y_1(t)dt\right) \\
&\qquad + \int_0^1 w^2(t)dt \\
&= -\beta^2 \int_0^1 Y_1^2(t)dt + \beta Y_1^2(1) + \int_0^1 w^2(t)dt\,,
\end{aligned}$$

which yields the last expression in (4.41).

2.4 The first equality is a consequence of Theorem 4.1. Since

$$\begin{aligned}
X &= X(1) = e^{\gamma}\int_0^1 e^{-\gamma s}dw(s)\,, \\
Y &= \int_0^1 e^{-\beta t}X(t)dt = \frac{1}{\beta-\gamma}\int_0^1 (e^{-\beta t} - e^{\gamma-\beta}e^{-\gamma t})dw(t)\,,
\end{aligned}$$

we obtain $(X,Y)' \sim \mathrm{N}(0,\Sigma)$, where

$$\begin{aligned}
\Sigma_{11} &= \frac{e^{2\gamma}-1}{2\gamma}, \qquad \Sigma_{12} = \frac{1}{\beta-\gamma}\left[\frac{e^{-\beta}(1-e^{2\gamma})}{2\gamma} - \frac{e^{-\beta}-e^{\gamma}}{\beta+\gamma}\right], \\
\Sigma_{22} &= \frac{1}{(\beta-\gamma)^2}\left[\frac{1-e^{-2\beta}}{2\beta} + \frac{2e^{-\beta}(e^{-\beta}-e^{\gamma})}{\beta+\gamma} + \frac{e^{-2\beta}(e^{2\gamma}-1)}{2\gamma}\right].
\end{aligned}$$

Then

$$m_1(\theta) = |I_2 - 2A\Sigma|^{-\frac{1}{2}} \exp\left(\frac{\beta+\gamma}{2}\right),$$

where $A_{11} = (-\beta-\gamma)/2$, $A_{12} = A_{21} = -\beta^2 e^{\beta}/2$ and $A_{22} = 0$. Some manipulations yield the last expression in (4.42).

2.5 The derivation is almost the same as in the solution to Problem 2.4. We arrive at

$$E\left[\exp\left\{\theta\left(xS(F_1) - U\right)\right\}\right] = |\, I_2 - 2B\Sigma\,|^{-\frac{1}{2}} \exp\left(\frac{\beta+\gamma}{2}\right),$$

where $B_{11} = (-\beta-\gamma)/2$, $B_{22} = -\theta e^{2\beta}/2$, $B_{12} = B_{21} = -\beta^2 e^{\beta}/2$, while Σ is defined in the solution to Problem 2.4. The last expression in (4.44) is obtained after some calculations.

2.6 Noting that

$$T(\hat{\rho}-1) = \frac{\dfrac{1}{T^3}\displaystyle\sum_{j=1}^{T} y_{j-1}(y_j - y_{j-1})}{\dfrac{1}{T^4}\displaystyle\sum_{j=1}^{T} y_{j-1}^2},$$

$$\mathcal{L}\left(\frac{1}{T^3}\sum_{j=1}^{T} y_{j-1}(y_j - y_{j-1})\,,\ \frac{1}{T^4}\sum_{j=1}^{T} y_{j-1}^2\right) \longrightarrow \mathcal{L}\left(\int_0^1 F_1(t)dF_1(t)\,,\ \int_0^1 F_1^2(t)dt\right),$$

we can establish (4.46) by the continuous mapping theorem.

2.7 Noting that

$$T^2(\hat{\beta}-\beta) = \frac{\dfrac{1}{T^2}\displaystyle\sum_{j=1}^{T} y_{1j}\varepsilon_{2j}}{\dfrac{1}{T^4}\displaystyle\sum_{j=1}^{T} y_{1j}^2},$$

$$\mathcal{L}\left(\frac{1}{T^2}\sum_{j=1}^{T} y_{1j}\varepsilon_{2j}\,,\ \frac{1}{T^4}\sum_{j=1}^{T} y_{1j}^2\right) \longrightarrow \mathcal{L}\left(\int_0^1 F_1(t)dw_2(t)\,,\ \int_0^1 F_1^2(t)dt\right)\,,$$

we can establish (4.50) by the continuous mapping theorem.

3.1 Since it holds that

$$\mathcal{L}\left(\int_0^1 w'(t)Hw(t)dt\right) = \mathcal{L}\left(\int_0^1 w'(t)\Lambda w(t)dt\right)\,,$$

where Λ is the diagonal matrix with the eigenvalues of H on diagonals, we have (4.58) because of (4.10) and the independence property of components of $\{w(t)\}$.

3.2 We have, from the matrix version of Ito's theorem,

$$\int_0^1 X(t)dX'(t) + \left(\int_0^1 X(t)dX'(t)\right)' = X(1)X'(1) - I_q\,.$$

Premultiplying A and taking the trace yield the conclusion.

3.3 It follows from (4.54) that

$$X(1) \sim \mathrm{N}\left(0, \frac{1}{2}A^{-1}(e^{2A} - I_q)\right)$$

so that

$$\begin{aligned} E(e^{\theta S_1}) &= \left|I_q + \frac{1}{2}(e^{2A} - I_q)\right|^{-\frac{1}{2}} \exp\left\{\frac{1}{2}\,\mathrm{tr}\,(A)\right\} \\ &= \prod_{j=1}^{q}\left[\frac{1}{2}\left\{\exp(2\sqrt{-2\theta\lambda_j}) + 1\right\}\exp(-\sqrt{-2\theta\lambda_j})\right]^{-\frac{1}{2}} \\ &= \prod_{j=1}^{q}(\cosh\sqrt{-2\theta\lambda_j})^{-\frac{1}{2}}\,, \end{aligned}$$

which gives (4.58).

3.4 The eigenvalues of H are $1/2$ and $-1/2$, which yields (4.61) because of (4.58).

3.5 When G is symmetric, it follows from (4.59) that $S_3 = [\,w'(1)Gw(1) - \mathrm{tr}\,(G)]/2$. Since $w(1) \sim \mathrm{N}(0, I_q)$, we have

$$E(e^{\theta S_3}) = \exp\left\{-\frac{\theta}{2}\,\mathrm{tr}\,(G)\right\}\prod_{j=1}^{q}(1 - \theta\lambda_j)^{-\frac{1}{2}}\,,$$

where λ_j's are the eigenvalues of G.

3.6 Noting that

$$S_4 \mid \{w_1(t)\} \sim \mathrm{N}\left(0, \int_0^1 w_1^2(t)dt\right),$$

we obtain, from (4.58),

$$\begin{aligned} E(e^{\theta S_4}) &= E\left[\exp\left\{\frac{\theta^2}{2}\int_0^1 w_1^2(t)dt\right\}\right] \\ &= (\cos\theta)^{-\frac{1}{2}}. \end{aligned}$$

3.7 Using the relation that

$$E\left(w_2(s)dw_2(t)\right) = \begin{cases} 0 & s < t \\ dt & s \geq t, \end{cases}$$

we obtain

$$\begin{aligned} \int_0^1\int_0^1 w_1(t)dw_1(s)E\left(w_2(s)dw_2(t)\right) &= \int_0^1\left\{\int_t^1 dw_1(s)\right\}w_1(t)dt \\ &= \int_0^1\left(w_1(1) - w_1(t)\right)w_1(t)dt, \end{aligned}$$

from which (4.65) follows.

3.8 Define $dX(t) = -\beta X(t)dt + dw_1(t)$ with $X(0) = 0$. Then we need to compute

$$\begin{aligned} E\left(e^{\theta S_5}\right) &= E\left[E\left[e^{\theta S_5}|\{w_1(t)\}\right]\right] \\ &= E\left[\exp\left\{\frac{\theta^2}{2}\int_0^1\left(w_1(t) - \frac{1}{2}w_1(1)\right)^2 dt\right\}\right] \\ &= E\left[\exp\left\{\frac{\theta^2 + 4\beta}{8}X^2(1) - \frac{\theta^2}{2}X(1)\int_0^1 X(t)dt - \frac{\beta}{2}\right\}\right], \end{aligned}$$

where $\beta = i\theta$ and $(X, Y)' \sim \mathrm{N}(0, \Sigma)$ with

$$X = X(1) = e^{-\beta}\int_0^1 e^{\beta s}dw_1(s),$$

$$Y = \int_0^1 X(t)dt = \frac{e^{-\beta}}{\beta}\int_0^1\left(e^{\beta} - e^{\beta s}\right)dw_1(s),$$

$$V(X) = \frac{1-e^{-2\beta}}{2\beta}\,, \quad \mathrm{Cov}(X,Y) = \frac{e^{-\beta}}{2\beta^2}(e^{-\beta}-2) + \frac{1}{2\beta^2}\,,$$

$$V(Y) = \frac{e^{-\beta}}{2\beta^3}(-e^{-\beta}+4) + \frac{1}{\beta^2} - \frac{3}{2\beta^3}\,.$$

Since $E(e^{\theta S_5}) = |I_2 - 2A\Sigma|^{-\frac{1}{2}} e^{-\frac{\beta}{2}}$, where

$$A = \begin{pmatrix} \dfrac{\theta^2+4\beta}{8} & -\dfrac{\theta^2}{4} \\ & \\ -\dfrac{\theta^2}{4} & 0 \end{pmatrix},$$

we obtain $E(e^{\theta S_5}) = \left(\cos\dfrac{\theta}{2}\right)^{-1}$ after some algebra.

Chapter 5.

1.1 Since

$$X(t) = e^{-\beta t}\int_0^t e^{\beta s}dw(s)\,,$$

we have

$$\begin{aligned}
\int_0^1 X^2(t)dt &= \int_0^1 e^{-2\beta t}\left\{\int_0^t\int_0^t e^{\beta(u+v)}dw(u)dw(v)\right\}dt \\
&= \int_0^1\int_0^1\left\{\int_{\max(u,v)}^1 e^{-2\beta t}dt\right\}e^{\beta(u+v)}dw(u)dw(v) \\
&= SS\frac{e^{-\beta|s-t|}-e^{-\beta(2-s-t)}}{2\beta}\,dw(s)dw(t)\,.
\end{aligned}$$

1.2 Putting $w_1(t) = w(t)$ we have, from (4.65)

$$\begin{aligned}
V\,[\,S_2\,|\,\{w(t)\}\,] &= \int_0^1 w^2(t)dt - w(1)\int_0^1 w(t)dt + \frac{1}{4}w^2(1) \\
&= \frac{1}{4}\int_0^1\int_0^1 [4(1-\max(s,t)) - 2(1-s+1-t)+1]\,dw(s)dw(t) \\
&= \int_0^1\int_0^1 \frac{1}{4}\,[\,1-2\,|s-t\,|\,]dw(s)dw(t)\,.
\end{aligned}$$

1.3 From the expression given above (5.7), we have

$$S_3 = \frac{1}{2}g(1)w^2(1) - \frac{1}{2}\int_0^1 \Big(g'(t)w^2(t) + g(t)\Big)\,dt\,,$$

where

$$\begin{aligned}\int_0^1 g'(t)w^2(t)dt &= \int_0^1\int_0^1 [\,g(1) - g(\max(s,t))\,]\,dw(s)dw(t)\\ &= g(1)w^2(1) - \int_0^1\int_0^1 g(\max(s,t))\,dw(s)dw(t)\,.\end{aligned}$$

Thus we can arrive at the right side of (5.7).

1.4 Note first that

$$S_{T5} = \frac{1}{T^2}\,\varepsilon' CMC'\varepsilon\,,$$

where $M = I_T - ee'/T$ with $e = (1,\cdots,1)'$ and C is defined in (1.3). Defining $y_j = y_{j-1} + \varepsilon_j$ with $y_0 = 0$, the FCLT and the continuous mapping theorem yield

$$\begin{aligned}\mathcal{L}\left(\frac{1}{T^2}\varepsilon' C' MC\varepsilon\right) &= \mathcal{L}\left(\frac{1}{T^2}\sum_{j=1}^{T}(y_j - \bar{y})^2\right)\\ &\longrightarrow \mathcal{L}\left(\int_0^1 w^2(t)dt - \left(\int_0^1 w(t)dt\right)^2\right).\end{aligned}$$

Thus $\mathcal{L}(S_{T5})$ also has the same limiting distribution as above.

2.1 Since $a_1(T)$ is the coefficient of λ in the expansion of $D_T(\lambda) = |I_T - \lambda K_T/T|$, (5.14) clearly holds. As for (5.15) we use the formula:

$$\frac{d^2 D_T(\lambda)}{d\lambda^2} = \frac{d}{d\lambda}\sum_{j=1}^{T} D_{jT}(\lambda)\,,$$

where $D_{jT}(\lambda)$ is the determinant of $I_T - \lambda K_T/T$ with the j-th column replaced by its derivative with respect to λ. Evaluating at $\lambda = 0$ leads to (5.15).

2.2 Defining

(S7) $$G(t) = \int_0^t g(s)ds\,, \qquad (G(0) = 0)\,,$$

we have

$$\begin{aligned}
&\int_0^1 \int_0^1 [\,1-\max(s,t)\,]\, g(s)g(t)dsdt \\
&\quad = \int_0^1 \left[\int_0^1 g(s)ds - t\int_0^t g(s)ds - \int_t^1 sg(s)ds\right] g(t)dt \\
&\quad = \int_0^1 \left(\int_t^1 G(s)ds\right) g(t)dt = \int_0^1 G^2(t)dt \ge 0\,.
\end{aligned}$$

The other cases can be proved similarly.

2.3 It follows from (5.21) that

$$\int_0^1 K(t,t)dt = \sum_{n=1}^\infty \frac{1}{\lambda_n}\int_0^1 f_n^2(t)dt = \sum_{n=1}^\infty \frac{1}{\lambda_n}\,.$$

We also have

$$\begin{aligned}
\int_0^1\int_0^1 K^2(s,t)dsdt &= \sum_{m=1}^\infty\sum_{n=1}^\infty \frac{1}{\lambda_m\lambda_n}\left(\int_0^1 f_m(s)f_n(s)ds\right)^2 \\
&= \sum_{n=1}^\infty \frac{1}{\lambda_n^2}\,.
\end{aligned}$$

3.1 Let us consider

$$S_N = \int_0^1\int_0^1 K_N(s,t)dw(s)dw(t)\,,\ K_N(s,t) = \sum_{n=1}^N \frac{1}{\lambda_n} f_n(s)f_n(t)\,.$$

Then Mercer's theorem ensures that $\underset{N\to\infty}{\text{l.i.m.}}\, S_N = S$, where S is defined in (5.24). Moreover we have

$$S_N = \int_0^1\int_0^1 \sum_{n=1}^N \frac{1}{\lambda_n} f_n(s)f_n(t)dw(s)dw(t) = \sum_{n=1}^N \frac{1}{\lambda_n}\left\{\int_0^1 f_n(t)dw(t)\right\}^2\,,$$

which leads us to the conclusion.

3.2 Following the second definition of $D(\lambda)$ in (5.26), we have

$$\begin{aligned}
D(\lambda) &= \sum_{n=0}^\infty \frac{(-1)^n\lambda^n}{n!}\int_0^1\cdots\int_0^1 \left|\begin{pmatrix} g(t_1)\\ \vdots \\ g(t_n)\end{pmatrix}(g(t_1),\cdots,g(t_n))\right| dt_1\cdots dt_n \\
&= 1-\lambda\int_0^1 g^2(t)dt\,.
\end{aligned}$$

3.3 We have only to show that (5.30) implies (5.10). We have

$$\begin{aligned}\int_0^1 [\,1-\max(s,t)\,]\,f(s)ds &= -\frac{1}{\lambda}\int_0^1 [\,1-\max(s,t)\,]\,f''(s)ds \\ &= -\frac{1}{\lambda}\left[f'(1)-tf'(t)-\int_t^1 sf''(s)ds\right] \\ &= \frac{1}{\lambda}\,f(t)\,,\end{aligned}$$

which implies (5.10).

3.4 The integral equation (5.10) with $K(s,t)=\min(s,t)-st$ is equivalent to $f(t)=c_1\cos\sqrt{\lambda}\,t+c_2\sin\sqrt{\lambda}\,t$ with $f(0)=f(1)=0$. Thus λ $(\neq 0)$ is an eigenvalue if and only if $\sin\sqrt{\lambda}=0$, from which we obtain $D(\lambda)=\sin\sqrt{\lambda}/\sqrt{\lambda}$ as the FD of K so that (5.34) results.

3.5 Defining $G(t)$ as in (S7) we obtain

$$\int_0^1\int_0^1 K(s,t)g(s)g(t)dsdt=\frac{1}{4}\int_0^1 (G(1)-2G(t))^2dt\geq 0\,.$$

3.6 We show that (5.36) implies (5.10). We obtain

$$\begin{aligned}\int_0^1 K(s,t)f(s)ds &= \frac{-1}{4\lambda}\int_0^1 [\,1-2\,|s-t|\,]\,f''(s)ds \\ &= \frac{-1}{4\lambda}\left[f'(1)-f'(0)-2\left\{\int_0^t (t-s)f''(s)ds\right.\right. \\ &\qquad\qquad \left.\left.+\int_t^1 (s-t)f''(s)ds\right\}\right] \\ &= \frac{1}{\lambda}\,f(t)\,.\end{aligned}$$

3.7 The integral equation (5.10) with $K(s,t)$ given in (5.39) yields $f(t)=c_1\cos\sqrt{\lambda}\,t+c_2\sin\sqrt{\lambda}\,t+c_3$, where $f(0)=f(1)$, $f'(0)=f'(1)$ and

$$f(0)=\lambda\left[\frac{c_3}{12}+\frac{1}{2}\int_0^1 (t^2-t)f(t)dt\right]\,.$$

Then we have $M(\lambda)c = 0$, where $c = (c_1, c_2, c_3)'$ and

$$\begin{aligned} |M(\lambda)| &= \begin{vmatrix} 1-\cos\sqrt{\lambda} & -\sin\sqrt{\lambda} & 0 \\ \sin\sqrt{\lambda} & 1-\cos\sqrt{\lambda} & 0 \\ \cos\sqrt{\lambda} - \dfrac{2\sin\sqrt{\lambda}}{\sqrt{\lambda}} - 1 & \sin\sqrt{\lambda} + \dfrac{2\cos\sqrt{\lambda}}{\sqrt{\lambda}} - \dfrac{2}{\sqrt{\lambda}} & -2 \end{vmatrix} \\ &= -8\left(\sin\frac{\sqrt{\lambda}}{2}\right)^2 . \end{aligned}$$

Thus the eigenvalues are given by $\lambda_n = 4n^2\pi^2$ $(n = 1, 2, \cdots)$. Since $\mathrm{rank}(M(\lambda_n)) = 1$ for each λ_n, the multiplicity of each eigenvalue is two. In fact $\int_0^1 K(t,t)dt = \frac{1}{12}$, while $\sum_{n=1}^{\infty} 1\Big/\left(4n^2\pi^2\right) = \frac{1}{24}$. Then we obtain the FD of K as in (5.40).

3.8 We show that (5.42) implies (5.10). We obtain

$$\begin{aligned} \int_0^1 K(s,t)f(s)ds &= -\frac{1}{\lambda}\int_0^1 [\,1 - \max(s,t) + b\,]\, f''(s)ds \\ &= -\frac{1}{\lambda}\,[\,bf'(1) + f(1) - f(t)\,]\,, \end{aligned}$$

where $f(1) = -bf'(1)$ since

$$f(1) = \lambda b \int_0^1 f(s)ds = -b\int_0^1 f''(s)ds = -bf'(1)\,.$$

3.9 When $b = 0$, it is clear that the zeros of $h(z)$ are all simple. Suppose that $b \neq 0$. If the zeros of $h(z)$ are not simple, it holds that $h(z) = h'(z) = 0$, that is, $\cos z - bz\sin z = 0$ and $bz\cos z + (b+1)\sin z = 0$. Then it follows that $b + 1 + b^2z^2 = 0$ so that $z = \pm\sqrt{-b-1}/b$ and $\sin^2(\sqrt{-b-1}/b) = -1/b$, from which we have contradiction. In fact $\sin^2(\sqrt{-b-1}/b) < -1/b$ for any real b.

3.10 The zeros of $h(z)$ satisfy $\tan z = 1/(bz)$, where z is real or purely imaginary. Suppose first that b is positive. Then no purely imaginary number $z = ix$ satisfies $\tan ix = 1/(bix) \Leftrightarrow \tanh x = -1/(bx)$, as can be seen from the graphs of $\tanh x$ and $-1/(bx)$ with $b > 0$. When b is negative, the graphs of $\tanh x$ and $-1/(bx)$ cross at two points $\pm a$, say, which yields the conclusion.

4.1 Putting $l(t) = t^m$ and $L(t) = \int_0^t l^2(s)ds$, we consider

$$\int_0^1 \int_0^1 \left\{ \int_{\max(s,t)}^1 l^2(u)du \right\} g(s)g(t)dsdt$$
$$= \int_0^1 \left[L(1)G(1) - L(t)G(t) - \int_t^1 L(s)g(s)ds \right] g(t)dt$$
$$= L(1)G^2(1) - 2\int_0^1 L(t)G(t)g(t)dt \,,$$

where $G(t)$ is defined in (S7). Here we have

$$\int_0^1 L(t)G(t)g(t)dt = \frac{1}{2}\left[L(1)G^2(1) - \int_0^1 G^2(t)l^2(t)dt \right] ,$$

which implies that the kernel appearing in (5.45) is positive definite.

4.2 It is easy to derive (5.47) from (5.10). Suppose that (5.47) holds. Then, using the two boundary conditions and noting that

$$\left(\frac{f'(t)}{t^{2m}} \right)' + \lambda f(t) = 0 \,,$$

$$t^{2m} f(t) = -\frac{1}{\lambda}\left(f''(t) - \frac{2m}{t} f'(t) \right) ,$$

we have

$$\lambda \int_0^1 [\, 1 - (\max(s,t))^{2m+1} \,] \, f(s)ds$$
$$= -\int_0^1 \left(\frac{f'(s)}{s^{2m}} \right)' ds + t^{2m+1}\int_0^t \left(\frac{f'(s)}{s^{2m}} \right)' ds + \int_t^1 (sf''(s) - 2mf'(s))\, ds$$
$$= -f'(1) + t^{2m+1}\frac{f'(t)}{t^{2m}} + f'(1) - tf'(t) - (2m+1)(f(1) - f(t))$$
$$= (2m+1)f(t) \,.$$

4.3 It is easy to see that $f'(t)$ takes the form:

$$f'(t) = c_1 \left(\frac{\sqrt{\lambda}}{2(m+1)} \right)^{\nu} \frac{2m+1}{\Gamma(\nu+1)}\, t^{2m} \, [\, 1 + t \times \{\text{polynomials in } t\} \,]$$

so that $f'(t)/t^{2m} \to 0$ as $t \to 0$ implies the first row of $M(\lambda)$. Since

$$f(1) = c_1 J_\nu\left(\frac{\sqrt{\lambda}}{m+1}\right) + c_2 J_{-\nu}\left(\frac{\sqrt{\lambda}}{m+1}\right) = 0\,,$$

we also have the second row of $M(\lambda)$.

4.4 Using (5.48) and (5.49) we obtain the general solution to (5.56) as

$$f(t) = t^{\frac{m}{2}}\left\{c_1 J_\nu\left(\frac{\sqrt{-2\lambda m}}{m+1}\,t^{\frac{m+1}{2}}\right) + c_2 J_{-\nu}\left(\frac{\sqrt{-2\lambda m}}{m+1}\,t^{\frac{m+1}{2}}\right)\right\}\,,$$

where $\nu = -m/(m+1)$. Then $f'(t)$ takes the form:

$$f'(t) = c_2\left(\frac{\sqrt{-2\lambda m}}{2(m+1)}\right)^{\frac{m}{m+1}} \frac{m t^{m-1}}{\Gamma\left(\dfrac{2m+1}{m+1}\right)}[\,1 + t \times \{\text{polynomials in } t\}\,]$$

so that $f'(t)/t^{m-1} \to 0$ as $t \to 0$ implies

$$\left(\frac{\sqrt{-2\lambda m}}{2(m+1)}\right)^{\frac{m}{m+1}} \frac{m}{\Gamma\left(\dfrac{2m+1}{m+1}\right)}\,c_2 = 0\,.$$

Since it holds that, when $c_2 = 0$,

$$f'(1) = -c_1\frac{\sqrt{-2\lambda m}}{2}\,J_{\nu+1}\left(\frac{\sqrt{-2\lambda m}}{m+1}\right)\,,$$

the other condition $f'(1) = mf(1)$ yields, after some algebra, the FD $D_2(\lambda)$ as in (5.57), where we have used the relation described in the problem.

4.5 By the definition of the Bessel function we have

$$\begin{aligned} D_2(\lambda) &= \Gamma(\nu)\left[\frac{1}{\Gamma(\nu)} + \frac{\dfrac{\lambda m}{2(m+1)^2}}{\Gamma(\nu+1)} + \frac{\left(\dfrac{\lambda m}{2(m+1)^2}\right)^2}{2!\Gamma(\nu+2)} + \cdots\right] \\ &= 1 + \frac{\lambda m}{2\nu(m+1)^2} + \sum_{k=2}^{\infty}\frac{\left(\dfrac{\lambda m}{2(m+1)^2}\right)^k}{k!(\nu+k-1)(\nu+k-2)\cdots\nu}\,, \end{aligned}$$

where $\nu = -m/(m+1)$. Then it is seen that $D_2(\lambda)$ reduces to $1 - \dfrac{\lambda}{2}$ when $m = 0$.

4.6 It can be shown that the integral equation (5.10) with $K = K_3$ is equivalent to

$$h''(t) + \lambda t^{2m} h(t) = 0\,, \qquad h(0) = h(1) = 0\,,$$

where $h(t) = t^{-m} f(t)$. The general solution is given by

$$h(t) = \sqrt{t}\left\{c_1 J_\nu\left(\frac{\sqrt{\lambda}}{m+1} t^{m+1}\right) + c_2 J_{-\nu}\left(\frac{\sqrt{\lambda}}{m+1} t^{m+1}\right)\right\},$$

where $\nu = 1/(2(m+1))$. The boundary condition $h(0) = 0$ implies $c_2 = 0$ and the other condition $h(1) = 0$ yields, when $c_2 = 0$, $J_\nu(\sqrt{\lambda}/(m+1))c_1 = 0$. Then we can obtain the FD $D_3(\lambda)$ of K_3 as in (5.60) so that the c.f. of U_3 is given by $(D_3(2i\theta))^{-\frac{1}{2}}$.

4.7 The integral equation (5.10) with the kernel appearing on the right side of (5.62) is equivalent to

$$h''(t) + \frac{2m+2}{2m+1}\frac{h'(t)}{t} + \frac{\lambda}{(2m+1)^2}\, t^{-\frac{2m}{2m+1}} h(t) = 0\,,$$

with the boundary conditions $h(1) = 0$ and $t^{1/(2m+1)} h(t) \to 0$ as $t \to 0$, where $h(t) = f(t)/t^{1/(2m+1)}$. The general solution is

$$h(t) = t^{-\frac{1}{2(2m+1)}}\left\{c_1 J_\nu\left(\frac{\sqrt{\lambda}}{m+1} t^{\frac{m+1}{2m+1}}\right) + c_2 J_{-\nu}\left(\frac{\sqrt{\lambda}}{m+1} t^{\frac{m+1}{2m+1}}\right)\right\},$$

where $\nu = 1/(2(m+1))$. Then the boundary condition $t^{1/(2m+1)} h(t) \to 0$ as $t \to 0$ implies $c_2 = 0$ so that we obtain the same FD $D_3(\lambda)$ from $h(1) = 0$.

4.8 Consider the integral equation (5.10) with $K = K_4$. We obtain

$$f'(t) t^{-2m} = \lambda\left[\int_t^1 f(s)ds - \int_0^1 s^{2m+1} f(s)ds\right],$$

from which it follows that

$$f''(t) - \frac{2m}{t} f'(t) + \lambda t^{2m} f(t) = 0\,, \qquad f(0) = f(1) = 0\,.$$

The general solution is given by

$$f(t) = t^{\frac{2m+1}{2}}\left\{c_1 J_\nu\left(\frac{\sqrt{\lambda}}{m+1} t^{m+1}\right) + c_2 J_{-\nu}\left(\frac{\sqrt{\lambda}}{m+1} t^{m+1}\right)\right\},$$

where $\nu = (2m+1)/(2(m+1))$. The boundary condition $f(0) = 0$ implies $c_2 = 0$ and thus we obtain the FD $D_4(\lambda)$ as in (5.64) from $f(1) = 0$. Thus the c.f. of U_4 is given by $(D_4(2i\theta))^{-\frac{1}{2}}$.

4.9 From the boundary conditions $f(0) = f(1) = 0$ and the first condition in (5.76) we obtain $M(\lambda)c = 0$, where $c = (a_1,\, c_1,\, c_2)'$ and

$$M(\lambda) = \begin{pmatrix} 0 & 1 & 0 \\ \frac{45}{4}\left(1 - \frac{6}{\lambda}\right) & \cos\sqrt{\lambda} & \sin\sqrt{\lambda} \\ -\frac{90}{7\lambda} & M_{33}(\lambda) & M_{34}(\lambda) \end{pmatrix},$$

with $M_{33}(\lambda)$ and $M_{34}(\lambda)$ defined in (5.78). Making use of REDUCE we obtain

$$\begin{aligned} |M(\lambda)| &= \frac{45}{7\lambda^4}\Big[35\sqrt{\lambda}(\lambda^2 - 12\lambda + 36)\cos\sqrt{\lambda} \\ &\qquad + (5\lambda^3 - 147\lambda^2 + 840\lambda - 1260)\sin\sqrt{\lambda}\Big]\,, \end{aligned}$$

which yields the FD $\tilde{D}_5(\lambda)$ in (5.79).

4.10 We are led to consider $f''(t) + \lambda f(t) = -4\lambda a_1 \cos 2\pi t$ with the boundary conditions $f(0) = f(1) = 0$, where

$$a_1 = \int_0^1 \sin^2 \pi s f(s) ds\,.$$

When $\lambda \neq 4\pi^2$, the general solution is given by

$$f(t) = c_1 \cos\sqrt{\lambda}\, t + c_2 \sin\sqrt{\lambda}\, t + \frac{4\lambda a_1}{4\pi^2 - \lambda}\cos 2\pi t$$

and we have $M(\lambda)c = 0$, where $c = (a_1,\, c_1,\, c_2)'$ and

$$M(\lambda) = \begin{pmatrix} \frac{4\lambda}{4\pi^2 - \lambda} & 1 & 0 \\ \frac{4\lambda}{4\pi^2 - \lambda} & \cos\sqrt{\lambda} & \sin\sqrt{\lambda} \\ \frac{-4\pi^2}{4\pi^2 - \lambda} & h_1(\lambda) & h_2(\lambda) \end{pmatrix},$$

$$h_1(\lambda) = \frac{2\pi^2 \sin\sqrt{\lambda}}{\sqrt{\lambda}(4\pi^2 - \lambda)}\,, \qquad h_2(\lambda) = \frac{2\pi^2(1-\cos\sqrt{\lambda})}{\sqrt{\lambda}(4\pi^2 - \lambda)}\,.$$

Therefore we obtain

$$|M(\lambda)| = \frac{-4\pi^2}{4\pi^2-\lambda}\left[\sin\sqrt{\lambda} + \frac{4\sqrt{\lambda}}{4\pi^2-\lambda}\,(1-\cos\sqrt{\lambda})\right].$$

When $\lambda = 4\pi^2$, the general solution is given by

$$f(t) = c_1\cos 2\pi t + c_2 \sin 2\pi t - 4\pi a_1 t \sin 2\pi t$$

and the three conditions yield $Nc = 0$ with $|N| = 0$ so that $\lambda = 4\pi^2$ is found to be an eigenvalue of multiplicity 1. Then we can obtain the FD $D_7(\lambda)$ as in (5.85). Note that $D_7(4\pi^2) = 0$.

4.11 We are led to consider

$$f''(t) + \lambda f(t) = -4\lambda a_1 \cos 2\pi t + 2\lambda a_2 \sin 2\pi t$$

with the boundary conditions $f(0) = f(1) = 0$ and

$$a_1 = \int_0^1 \sin^2 \pi s\, f(s) ds\,, \qquad a_2 = \int_0^1 \sin 2\pi s\, f(s)ds\,.$$

When $\lambda \neq 4\pi^2$, the general solution is given by

$$f(t) = c_1 \cos\sqrt{\lambda}\,t + c_2\sin\sqrt{\lambda}\,t + \frac{4\lambda a_1}{4\pi^2-\lambda}\,\cos 2\pi t - \frac{2\lambda a_2}{4\pi^2-\lambda}\,\sin 2\pi t$$

and we have $M(\lambda)c = 0$, where $c = (a_1,\, a_2,\, c_1,\, c_2)'$ and

$$M(\lambda) = \begin{pmatrix} \dfrac{4\lambda}{4\pi^2-\lambda} & 0 & 1 & 0 \\[2ex] \dfrac{4\lambda}{4\pi^2-\lambda} & 0 & \cos\sqrt{\lambda} & \sin\sqrt{\lambda} \\[2ex] \dfrac{-4\pi^2}{4\pi^2-\lambda} & 0 & h_1(\lambda) & h_2(\lambda) \\[2ex] 0 & \dfrac{-4\pi^2}{4\pi^2-\lambda} & \dfrac{\sqrt{\lambda}}{\pi}h_2(\lambda) & -\dfrac{\sqrt{\lambda}}{\pi}h_1(\lambda) \end{pmatrix}$$

with $h_1(\lambda)$ and $h_2(\lambda)$ defined in the solution to Problem 4.10. Therefore we obtain

$$|M(\lambda)| = \left(\frac{4\pi^2}{4\pi^2-\lambda}\right)^2 \left[\sin\sqrt{\lambda} + \frac{4\sqrt{\lambda}}{4\pi^2-\lambda}\,(1-\cos\sqrt{\lambda})\right].$$

When $\lambda = 4\pi^2$, the general solution is given by

$$f(t) = c_1\cos 2\pi t + c_2\sin 2\pi t - 4\pi a_1 t\sin 2\pi t - 2\pi a_2 t\cos 2\pi t\,.$$

The four conditions above yield $Nc = 0$, where $c = (a_1,\, a_2,\, c_1,\, c_2)'$ and

$$N = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -2\pi & 1 & 0 \\ -\dfrac{1}{4} & \dfrac{\pi}{4} & -\dfrac{1}{4} & 0 \\ -\pi & -\dfrac{3}{4} & 0 & \dfrac{1}{2} \end{pmatrix}, \qquad |N| = -\frac{\pi}{4}\,.$$

Thus $\lambda = 4\pi^2$ cannot be an eigenvalue. Then we obtain the FD $D_8(\lambda)$ as in (5.87). Note that $D_8(4\pi^2) \neq 0$.

5.1 Noting that $Z_n \sim \mathrm{N}(0,1)$ we have

$$\begin{aligned} &E\left[\exp\left\{\frac{i\theta}{\lambda_n}\left(Z_n^2 + af_n(0)Z_n\right)\right\}\right] \\ &\quad = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\left\{x^2 - \frac{2i\theta}{\lambda_n}\left(x^2 + af_n(0)x\right)\right\}\right]dx \\ &\quad = \left(1-\frac{2i\theta}{\lambda_n}\right)^{-\frac{1}{2}}\exp\left\{\frac{(ia\theta)^2}{2}\,\frac{f_n^2(0)}{\lambda_n(\lambda_n-2i\theta)}\right\}, \end{aligned}$$

which yields (5.102).

5.2 Using the second relation in (5.98) and Mercer's theorem we have

$$\begin{aligned} \sum_{n=1}^{\infty}\frac{f_n^2(0)}{\lambda_n(\lambda_n-2i\theta)} &= \frac{1}{2i\theta}\left[\sum_{n=1}^{\infty}\frac{f_n^2(0)}{\lambda_n-2i\theta} - \sum_{n=1}^{\infty}\frac{f_n^2(0)}{\lambda_n}\right] \\ &= \frac{1}{2i\theta}\left\{\Gamma(0,0;2i\theta) - K(0,0)\right\}, \end{aligned}$$

which proves the theorem.

5.3 That (5.104) implies (5.105) is easily established. Suppose that (5.105) holds. Using the two boundary conditions in (5.105) and noting that

$$\left(\frac{h'(t)}{t^{2m}}\right)' + \lambda h(t) = 0\,, \quad t^{2m}h(t) = -\frac{1}{\lambda}\left(h''(t) - \frac{2m}{t}\,h'(t)\right)\,,$$

we can show that

$$\lambda \int_0^1 \left[1 - (\max(s,t))^{2m+1}\right] h(s)ds = t^{2m+1} - 1 + (2m+1)h(t)\,.$$

5.4 We have only to derive the resolvent $\Gamma(s,t;\lambda)$ of $K(s,t) = (\max(s,t))^m/2$ evaluated at the origin. Putting $h(t) = \Gamma(0,t;\lambda)$ we consider

$$\begin{aligned} h(t) &= K(0,t) + \lambda \int_0^t h(s)K(s,t)ds \\ &= \frac{1}{2}\,t^m + \lambda\left[\frac{1}{2}\,t^m \int_0^t h(s)ds + \frac{1}{2}\int_t^1 s^m h(s)ds\right]\,, \end{aligned}$$

which is equivalent to

$$h''(t) - \frac{m-1}{t}\,h'(t) - \frac{\lambda m}{2}\,t^{m-1}h(t) = 0\,, \quad \lim_{t\to 0}\frac{h'(t)}{t^{m-1}} = \frac{m}{2}\,, \quad h'(1) = mh(1)\,.$$

The general solution is given by

$$h(t) = t^{\frac{m}{2}}\left\{c_1 J_\nu\left(\frac{\sqrt{-2\lambda m}}{m+1}t^{\frac{m+1}{2}}\right) + c_2 J_{-\nu}\left(\frac{\sqrt{-2\lambda m}}{m+1}t^{\frac{m+1}{2}}\right)\right\}\,,$$

where $\nu = -m/(m+1)$. From the two boundary conditions we can determine c_1 and c_2 uniquely. Then

$$\begin{aligned} \Gamma(0,0;\lambda) &= h(0) = \frac{c_1}{\Gamma(\nu+1)}\left(\frac{\sqrt{-2\lambda m}}{2(m+1)}\right)^\nu \\ &= \frac{1}{2}\frac{\Gamma(-\nu+1)}{\Gamma(\nu+1)}\,\frac{J_{-\nu+1}\left(\dfrac{\sqrt{-2\lambda m}}{m+1}\right)}{J_{\nu-1}\left(\dfrac{\sqrt{-2\lambda m}}{m+1}\right)}\,\left(\frac{\sqrt{-2\lambda m}}{2(m+1)}\right)^{2\nu} \end{aligned}$$

so that (5.111) is established by Theorem 5.8 and (5.57).

5.5 It is easy to deduce that

$$\begin{aligned}
\int_0^1 q(t)r(t)dt &= \int_0^1 \sum_{n=1}^{\infty} \frac{c_n}{\sqrt{\lambda_n}} f_n(t) \left(m(t) - \sum_{m=1}^{\infty} \frac{c_m}{\sqrt{\lambda_m}} f_m(t) \right) dt \\
&= \sum_{n=1}^{\infty} \frac{c_n^2}{\lambda_n} - \sum_{m,n=1}^{\infty} \frac{c_m}{\sqrt{\lambda_m}} \frac{c_n}{\sqrt{\lambda_n}} \int_0^1 f_m(t) f_n(t) dt \\
&= \sum_{n=1}^{\infty} \frac{c_n^2}{\lambda_n} - \sum_{n=1}^{\infty} \frac{c_n^2}{\lambda_n} = 0\,.
\end{aligned}$$

5.6 We first obtain

$$E\left[\exp\left\{\frac{i\theta}{\lambda_n}(Z_n + c_n)^2\right\}\right] = \left(1 - \frac{2i\theta}{\lambda_n}\right)^{-\frac{1}{2}} \exp\left\{\frac{i\theta c_n^2}{\lambda_n - 2i\theta}\right\}$$

so that

$$\begin{aligned}
E\left(e^{i\theta S_Z}\right) &= \prod_{n=1}^{\infty} \left(1 - \frac{2i\theta}{\lambda_n}\right)^{-\frac{1}{2}} \exp\left\{\sum_{n=1}^{\infty} \frac{i\theta c_n^2}{\lambda_n - 2i\theta} + i\theta \int_0^1 r^2(t)dt\right\} \\
&= (D(2i\theta))^{-\frac{1}{2}} \exp\left\{i\theta \int_0^1 m^2(t)dt + i\theta \sum_{n=1}^{\infty} \left(\frac{c_n^2}{\lambda_n - 2i\theta} - \frac{c_n^2}{\lambda_n}\right)\right\} \\
&= (D(2i\theta))^{-\frac{1}{2}} \exp\left\{i\theta \int_0^1 m^2(t)dt - 2\theta^2 \sum_{n=1}^{\infty} \frac{c_n^2}{\lambda_n(\lambda_n - 2i\theta)}\right\}
\end{aligned}$$

5.7 The integral equation (5.117) with $K(s,t) = \mathrm{Cov}(w(s) - sw(1), w(t) - tw(1)) = \min(s,t) - st$ and $m(s) = a + bs$ is equivalent to $h''(t) + \lambda h(t) = -a - bt$ with $h(0) = h(1) = 0$, where the general solution is given by $h(t) = c_1 \cos\sqrt{\lambda}\, t + c_2 \sin\sqrt{\lambda}\, t - (a + bt)/\lambda$. From the boundary conditions we have

$$c_1 = \frac{a}{\lambda}\,, \qquad c_2 = \frac{a+b}{\lambda \sin\sqrt{\lambda}} - \frac{a}{\lambda} \cot\sqrt{\lambda}\,.$$

Then

$$\begin{aligned}
&\frac{\lambda}{2} \int_0^1 \{m^2(t) + \lambda h(t) m(t)\} dt \\
&\quad = \frac{1}{\cos\sqrt{\lambda} + 1} \left[a(a+b)\sqrt{\lambda} \sin\sqrt{\lambda} + \frac{b^2}{2} \left(\frac{\sqrt{\lambda} \cos\sqrt{\lambda} \sin\sqrt{\lambda}}{\cos\sqrt{\lambda} - 1} + \cos\sqrt{\lambda} + 1\right)\right],
\end{aligned}$$

which yields the c.f. given in (5.123).

5.8 The integral equation (5.117) with $K = K_6$ is equivalent to

$$h''(t) + \lambda h(t) = -\frac{a}{2\pi}(1 - \cos 2\pi t) + 2\lambda c_3 \sin 2\pi t\,, \qquad h(0) = h(1) = 0$$

and

$$c_3 = \int_0^1 h(s) \sin 2\pi s ds\,.$$

The general solution is

$$h(t) = c_1 \cos\sqrt{\lambda}\,t + c_2 \sin\sqrt{\lambda}\,t - \frac{a}{2\lambda\pi} + \frac{a}{2\pi}\frac{\cos 2\pi t}{\lambda - 4\pi^2} + 2\lambda c_3 \frac{\sin 2\pi t}{\lambda - 4\pi^2}\,.$$

From the above conditions we have

$$c_1 = \frac{2\pi a}{\lambda(4\pi^2 - \lambda)}\,, \qquad c_2 = \frac{c_1(1 - \cos\sqrt{\lambda})}{\sin\sqrt{\lambda}}$$

and thus

$$\begin{aligned}
&\frac{\lambda}{2}\int_0^1 \left\{\frac{a^2(1 - \cos 2\pi t)^2}{4\pi^2} + \lambda h(t)\frac{a}{2\pi}(1 - \cos 2\pi t)\right\} dt \\
&\quad = a^2\left[\frac{\lambda}{4(4\pi^2 - \lambda)} + \frac{4\pi^2\sqrt{\lambda}}{(4\pi^2 - \lambda)^2}\,\frac{1 - \cos\sqrt{\lambda}}{\sin\sqrt{\lambda}}\right]\,,
\end{aligned}$$

which yields (5.125).

5.9 Let us define

$$S_N = \sum_{n=1}^{N} \frac{1}{\lambda_n}\left(Z_n^2 + a f_n(0) Z_n Z\right) + bZ^2 = W'AW\,,$$

where $W = (Z_1, \cdots, Z_N, Z)'$ and

$$\begin{aligned}
A &= \begin{pmatrix} \Lambda & h \\ h' & b \end{pmatrix}\,, \qquad \Lambda = \mathrm{diag}\left(\frac{1}{\lambda_1}, \cdots, \frac{1}{\lambda_N}\right)\,, \\
h &= \left(\frac{a f_1(0)}{2\lambda_1}, \cdots, \frac{a f_N(0)}{2\lambda_N}\right)'\,.
\end{aligned}$$

Since

$$
\begin{aligned}
E\left(e^{i\theta S_N}\right) &= |I_{N+1} - 2i\theta A|^{-\frac{1}{2}} \\
&= \left[|I_N - 2i\theta\Lambda|\left\{1 - 2ib\theta + 4\theta^2 h'(I_N - 2i\theta\Lambda)^{-1}h\right\}\right]^{-\frac{1}{2}} \\
&= \left[\prod_{n=1}^{N}\left(1 - \frac{2i\theta}{\lambda_n}\right)\right]^{-\frac{1}{2}} \left[1 - 2ib\theta + a^2\theta^2 \sum_{n=1}^{N} \frac{f_n^2(0)}{\lambda_n(\lambda_n - 2i\theta)}\right]^{-\frac{1}{2}}
\end{aligned}
$$

and S_N converges in probability to S as $N \to \infty$, (5.128) is established.

5.10 Using the definition of $m(t) = q(t) + r(t)$ with $q(t)$ defined below (5.112) we have

$$
\begin{aligned}
S_Z &= \int_0^1 \left\{\sum_{n=1}^{\infty} \frac{f_n(t)}{\sqrt{\lambda_n}} Z_n + (q(t) + r(t))Z\right\}^2 dt \\
&= \int_0^1 \left\{\sum_{n=1}^{\infty} \frac{f_n(t)}{\sqrt{\lambda_n}} (Z_n + c_n Z)^2\right\} dt + Z^2 \int_0^1 r^2(t)dt \\
&\qquad\qquad +2Z \int_0^1 r(t) \left(\sum_{n=1}^{\infty} \frac{f_n(t)}{\sqrt{\lambda_n}} Z_n + q(t)Z\right) dt \\
&= \sum_{n=1}^{\infty} \frac{1}{\lambda_n} (Z_n + c_n Z)^2 + Z^2 \int_0^1 r^2(t)dt \,,
\end{aligned}
$$

where use has been made of the facts that

$$
\int_0^1 f_m(t) f_n(t) dt = \delta_{mn} \,, \qquad \int_0^1 r(t)q(t) = 0 \,,
$$

$$
\int_0^1 r(t) f_n(t) dt = \int_0^1 (m(t) - q(t)) f_n(t) dt = 0 \,.
$$

5.11 Defining

$$
S_N = \sum_{n=1}^{N} \frac{1}{\lambda_n} (Z_n + c_n Z)^2 + Z^2 \int_0^1 r^2(t)dt = W'AW \,,
$$

where $W = (Z_1, \cdots, Z_N, Z)'$ and

$$
\begin{aligned}
A &= \begin{pmatrix} \Lambda & h \\ h' & \gamma \end{pmatrix}, \qquad \Lambda = \mathrm{diag}\left(\frac{1}{\lambda_1}, \cdots, \frac{1}{\lambda_N}\right), \\
h &= \left(\frac{c_1}{\lambda_1}, \cdots, \frac{c_N}{\lambda_N}\right)', \qquad \gamma = \int_0^1 r^2(t)dt + \sum_{n=1}^{N} \frac{c_n^2}{\lambda_n} \,,
\end{aligned}
$$

we obtain

$$\begin{aligned} E\left(e^{i\theta S_N}\right) &= |I_{N+1} - 2i\theta A|^{-\frac{1}{2}} \\ &= \left[\prod_{n=1}^{N}\left(1-\frac{2i\theta}{\lambda_n}\right)\right]^{-\frac{1}{2}}\left[1-2i\theta\gamma+4\theta^2\sum_{n=1}^{N}\frac{c_n^2}{\lambda_n(\lambda_n-2i\theta)}\right]^{-\frac{1}{2}}. \end{aligned}$$

Noting that, as $N \to \infty$,

$$\gamma \longrightarrow \int_0^1 r^2(t)dt + \sum_{n=1}^{\infty}\frac{c_n^2}{\lambda_n} = \int_0^1 m^2(t)dt\,,$$

we have

$$E\left(e^{i\theta S}\right) = (D(2i\theta))^{-\frac{1}{2}}\left[1-2i\theta\int_0^1 m^2(t)dt + 4\theta^2\sum_{n=1}^{\infty}\frac{c_n^2}{\lambda_n(\lambda_n-2i\theta)}\right]^{-\frac{1}{2}}.$$

Since

$$\frac{c_n^2}{\lambda_n} = \int_0^1\int_0^1 m(s)m(t)f_n(s)f_n(t)dsdt\,,$$

we can establish Theorem 5.11 using the second relation for the resolvent in (5.98).

6.1 Since $\log(1-(\beta/T)) = -\beta/T + O(T^{-2})$, we have

$$\begin{aligned} \left|B_T(j,k) - K\left(\frac{j}{T},\frac{k}{T}\right)\right| &= \left|\exp\left\{|\,j-k\,|\log\left(1-\frac{\beta}{T}\right)\right\} - \exp\left\{-\frac{\beta}{T}|\,j-k\,|\right\}\right| \\ &= \exp\left\{-\frac{\beta}{T}\,|\,j-k\,|\right\}\left|\exp\left\{|\,j-k\,|\,O(T^{-2})\right\}-1\right|, \end{aligned}$$

which evidently goes to 0 uniformly for all j, k as $T \to \infty$.

6.2 Putting $d_T(j,k) = B_T(j,k) - K(j/T,k/T)$ and $\delta_T = \max|d_T(j,k)|$ we have

$$\begin{aligned} R_T &= \frac{1}{T}\sum_{j,k=1}^{T} d_T(j,k)\varepsilon_j\varepsilon_k \\ &= \frac{1}{T}\sum_{j=1}^{T} d_T(j,j)\varepsilon_j^2 + \frac{1}{T}\sum_{j\neq k} d_T(j,k)\varepsilon_j\varepsilon_k \\ &= Q_1 + Q_2\,, \end{aligned}$$

where

$$E(\,|Q_1\,|\,) \le \delta_T\,, \qquad E\left(Q_2^2\right) \le \frac{2T(T-1)}{T^2}\,\delta_T^2 \le 2\delta_T^2\,.$$

Thus we have $E(\,|\,R_T\,|\,) \le (1+\sqrt{2})\delta_T \to 0$ so that the conclusion follows from Markov's inequality.

6.3 We consider, putting $s = j/T$ and $t = k/T$,

$$\begin{aligned}
&\left|\frac{\rho^{|j-k|}-\rho^{2T-j-k+2}}{T(1-\rho^2)} - \frac{e^{-\beta|s-t|}-e^{-\beta(2-s-t)}}{2\beta}\right| \\
&\qquad \le \left|\frac{\rho^{|j-k|}}{T(1-\rho^2)} - \frac{e^{-\beta|s-t|}}{2\beta}\right| + \left|\frac{\rho^{2T-j-k+2}}{T(1-\rho^2)} - \frac{1}{2\beta}\,e^{-\beta(2-s-t)}\right| .
\end{aligned}$$

The quantities on the right side can be shown to converge to 0 uniformly for all j, k as $T\to\infty$, as in the solution to Problem 6.1.

6.4 Put $B_T = \Omega^{-1}/T = (\!(B_T(j,k))\!)$ and consider the absolute difference of the (j,k)-th element of Ω^{-2}/T^3 and $K_{(2)}(j/T,\,k/T)$, which is

$$\begin{aligned}
&\left|\frac{1}{T}\sum_{l=1}^{T} B_T(j,l)B_T(l,k) - \int_0^1 K\left(\frac{j}{T},u\right)K\left(u,\frac{k}{T}\right)du\right| \\
&\quad = \left|\sum_{l=1}^{T}\int_{\frac{l-1}{T}}^{\frac{l}{T}}\left\{B_T(j,l)B_T(l,k) - K\left(\frac{j}{T},u\right)K\left(u,\frac{k}{T}\right)\right\}du\right| \\
&\quad \le \sum_{l=1}^{T}\int_{\frac{l-1}{T}}^{\frac{l}{T}}\left|\left\{B_T(j,l) - K\left(\frac{j}{T},u\right)\right\}B_T(l,k)\right. \\
&\qquad\qquad\qquad \left. + K\left(\frac{j}{T},u\right)\left\{B_T(l,k) - K\left(u,\frac{k}{T}\right)\right\}\right|du\,.
\end{aligned}$$

This last quantity is shown to converge to 0 uniformly for all j, k as $T\to\infty$. Then we consider, in stead of S_{T4},

$$\begin{aligned}
S'_{T4} &= \frac{1}{T}\sum_{j,k=1}^{T}\left[K\left(\frac{j}{T},\frac{k}{T}\right) + \gamma K_{(2)}\left(\frac{j}{T},\frac{k}{T}\right)\right]\varepsilon_j\varepsilon_k \\
&= \sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n}+\frac{\gamma}{\lambda_n^2}\right)\left(\frac{1}{\sqrt{T}}\sum_{j=1}^{T} f_n\left(\frac{j}{T}\right)\varepsilon_j\right)^2 ,
\end{aligned}$$

which yields the first expression in (5.150). The distributional equivalence of the first and second expressions is obvious.

6.5 We have only to establish (5.152), which is easily proved from (5.151) and the definition of the FD given in (5.25).

6.6 We can show easily that

$$R_T = \frac{1}{T}\sum_{j,k=1}^{T}\left\{B_T(j,k) - K\left(\frac{j}{T},\frac{k}{T}\right)\right\}\underset{\sim}{\varepsilon}_j' H \underset{\sim}{\varepsilon}_k$$

converges in probability to 0. Thus we consider

$$\frac{1}{T}\sum_{j,k=1}^{T} K\left(\frac{j}{T},\frac{k}{T}\right)\underset{\sim}{\varepsilon}_j' H \underset{\sim}{\varepsilon}_k = \sum_{n=1}^{\infty}\frac{1}{\lambda_n}\frac{1}{\sqrt{T}}\sum_{j=1}^{T} f_n\left(\frac{j}{T}\right)\underset{\sim}{\varepsilon}_j' H \frac{1}{\sqrt{T}}\sum_{k=1}^{T} f_n\left(\frac{k}{T}\right)\underset{\sim}{\varepsilon}_k\,,$$

which converges in distribution to $\sum_{n=1}^{\infty} Z_n' H Z_n/\lambda_n$, where $\{Z_n\} \sim \mathrm{NID}(0, I_q)$. Then Mercer's theorem establishes (5.154).

6.7 Noting that

$$\begin{aligned}\mathcal{L}\left(\int_0^1\int_0^1 K(s,t)dw'(s)Hdw(t)\right) &= \mathcal{L}\left(\sum_{n=1}^{\infty}\frac{1}{\lambda_n}Z_n' H Z_n\right)\\ &= \mathcal{L}\left(\sum_{n=1}^{\infty}\frac{1}{\lambda_n}Z_n'\Lambda Z_n\right),\end{aligned}$$

where $\Lambda = \mathrm{diag}(\delta_1,\cdots,\delta_q)$, we obtain, as the c.f. of this last distribution,

$$\prod_{n=1}^{\infty}\left\{\prod_{j=1}^{q}\left(1-\frac{2i\delta_j\theta}{\lambda_n}\right)^{-\frac{1}{2}}\right\} = \prod_{j=1}^{q}(\,D(2i\delta_j\theta))^{-\frac{1}{2}}\,.$$

6.8 Put $B_N = C'(\rho)C(\rho)/N = (\!(B_N(j,k)\,)\!)$. Then we have

$$\frac{1}{N^2}y'y = \frac{1}{N}\sum_{j,k=1}^{T} B_N(j,k)\underset{\sim}{\varepsilon}_j'\underset{\sim}{\varepsilon}_k\,,$$

where $\underset{\sim}{\varepsilon}_j = (\varepsilon_{(j-1)m+1},\cdots,\varepsilon_{jm})' \; : \; m\times 1$. Thus (5.157) follows from (5.148) and (5.154).

6.9 Putting $d_T(j,k) = B_T(j,k) - K(j/T,k/T)$ and $\delta_T = \max|d_T(j,k)|$ we have

$$R_T = \sum_{l,m=0}^{\infty}(\,Q_1(l,m) + Q_2(l,m)\,)\,,$$

where

$$Q_1(l,m) = \frac{\alpha_l\alpha_m}{T}\sum_{j=\max(1-l,1-m)}^{\min(T-l,T-m)} d_T(j+l,\,j+m)\varepsilon_j^2\,,$$

$$Q_2(l,m) = \frac{\alpha_l \alpha_m}{T} \sum_{j=1-l}^{T-l} \sum_{\substack{k=1-m \\ j \neq k}}^{T-m} d_T(j+l,\ k+m) \varepsilon_j \varepsilon_k \,.$$

Then we can establish that

$$\begin{aligned} E(\,|Q_1|\,) &\leq c_1 \delta_T |\alpha_l|\,|\alpha_m|\,, \\ E(\,Q_2^2\,) &\leq c_2\,(\delta_T |\alpha_l|\,|\alpha_m|\,)^2\,, \end{aligned}$$

for some positive constants c_1 and c_2. Therefore, by Schwarz's and Markov's inequalities, we see that, for every $x > 0$,

$$P\,(\,|R_T| > x\,) \leq \frac{c_1 + \sqrt{c_2}}{x}\,\delta_T \left(\sum_{l=0}^{\infty} |\alpha_l| \right)^2 \longrightarrow 0\,.$$

6.10 We consider

$$\begin{aligned} V_T' &= \frac{1}{T} \sum_{j,k=1}^{T} K\left(\frac{j}{T}, \frac{k}{T} \right) u_j u_k \\ &= \sum_{l,m=0}^{\infty} \alpha_l \alpha_m \frac{1}{T} \sum_{j,k=1}^{T} K\left(\frac{j}{T}, \frac{k}{T} \right) \varepsilon_{j-l}\, \varepsilon_{k-m} \\ &= V_{T,M}' + R_{T,M}\,, \end{aligned}$$

where

$$V_{T,M}' = \sum_{l,m=0}^{M} \alpha_l \alpha_m \frac{1}{T} \sum_{j,k=1}^{T} K\left(\frac{j}{T}, \frac{k}{T} \right) \varepsilon_{j-l}\, \varepsilon_{k-m}$$

and $R_{T,M}$ is the remainder term. There exists a sequence $\{a_M\}$ such that $E(\,|R_{T,M}|\,) \leq a_M$ for all T and $a_M \to 0$ as $M \to \infty$. We further deduce that, for M fixed,

$$\mathcal{L}(V_{T,M}') \longrightarrow \mathcal{L}\left(\left(\sum_{l=0}^{M} \alpha_l \right)^2 \int_0^1 \int_0^1 K(s,t) dw(s) dw(t) \right)\,,$$

which yields (5.161).

Chapter 6.

3.1 It is easy to deduce that, for $x \geq 0$,

$$
\begin{aligned}
P(X^2 - Y^2 \leq x) &= \int_{-\infty}^{\infty} \left\{ \int_{-\sqrt{x+t^2}}^{\sqrt{x+t^2}} \frac{1}{\sqrt{2\pi}} e^{-s^2/2} ds \right\} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \\
&= \int_{-\infty}^{\infty} \left\{ 1 - 2\Phi(-\sqrt{x+t^2}) \right\} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \\
&= 1 - \frac{4}{\sqrt{2\pi}} \int_0^{\infty} \Phi(-\sqrt{x+t^2}) e^{-t^2/2} dt.
\end{aligned}
$$

3.2 Lévy's inversion formula (6.1) yields

$$
\begin{aligned}
\mathrm{Re}\left[\frac{1 - e^{-i\theta x}}{i\theta} \phi_3(\theta) \right] &= \mathrm{Re}\left[\frac{\sin\theta x - i(1 - \cos\theta x)}{\theta} \phi_3(\theta) \right] \\
&= \frac{1}{\theta} \left[\mathrm{Re}\{\phi_3(\theta)\} \sin\theta x + \mathrm{Im}\{\phi_3(\theta)\}(1 - \cos\theta x) \right]
\end{aligned}
$$

and we obtain the second equality in (6.21) by transforming θ into $\theta = u^4$.

3.3 Consider

$$
\begin{aligned}
\sum_{k=0}^{\infty} (-1)^{k+N} V_{k+N} &= (-1)^N (V_N - V_{N+1} + V_{N+2} - \cdots) \\
&= (-1)^N (1 - F + F^2 - \cdots) V_N \\
&= (-1)^N \frac{V_N}{1+F}.
\end{aligned}
$$

Since

$$
\begin{aligned}
\frac{1}{1+x} &= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k}{dx^k} \left(\frac{1}{1+x} \right) \bigg|_{x=1} (x-1)^k \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}} (x-1)^k,
\end{aligned}
$$

we can establish the last equality in (6.26).

4.1 Defining an auxiliary process $dX(t) = -\beta X(t)dt + dw(t)$ with $X(0) = 0$, we have

$$
\begin{aligned}
E[\exp\{\theta(xV_6 - U_6)\}] &= E\left[\exp\left\{ \theta x \int_0^1 w^2(t)dt - \theta \int_0^1 w(t)dw(t) \right\} \right] \\
&= E\left[\exp\left\{ \theta x \int_0^1 X^2(t)dt - \theta \int_0^1 X(t)dX(t) \right\} \frac{d\mu_w}{d\mu_X}(X) \right] \\
&= E\left[\exp\left\{ \frac{\beta - \theta}{2} X^2(1) \right\} \right] \exp\left(\frac{\theta - \beta}{2} \right),
\end{aligned}
$$

where $\beta = \sqrt{-2\theta x}$. Since $X(1) \sim \mathrm{N}(0, (1-e^{-2\beta})/(2\beta))$, it holds that

$$\begin{aligned} E\left[\exp\left\{\frac{\beta-\theta}{2}X^2(1)\right\}\right] e^{-\beta/2} &= \left[1 - \frac{(\beta-\theta)(1-e^{-2\beta})}{2\beta}\right]^{-\frac{1}{2}} e^{-\beta/2} \\ &= \left(\cosh\beta + \frac{\theta}{\beta}\sinh\beta\right)^{-\frac{1}{2}}, \end{aligned}$$

which yields (6.40).

4.2 Noting that $\mathrm{Im}[\phi_6(0;x)] = 0$ we obtain

$$\begin{aligned} \lim_{u\to 0} \mathrm{Im}\left[\frac{\phi_6(u^2;x)}{u}\right] &= \mathrm{Im}\left[\lim_{u\to 0}\left\{\frac{\phi_6(u^2;x)}{u^2} \times u\right\}\right] \\ &= \mathrm{Im}\left[\left.\frac{\partial\phi_6(\theta;x)}{\partial\theta}\right|_{\theta=0} \times \lim_{u\to 0} u\right] \\ &= 0, \end{aligned}$$

where $\partial\phi_6(\theta;x)/\partial\theta|_{\theta=0}$ is given in the solution to Problem 4.3.

4.3 Proceeding in the same way as in the solution to Problem 4.2, we have

$$\lim_{\theta\to 0}\frac{1}{\theta}\mathrm{Im}\left[\phi_6(\theta;x)\right] = \mathrm{Im}\left[\left.\frac{\partial\phi_6(\theta;x)}{\partial\theta}\right|_{\theta=0}\right],$$

where

$$\begin{aligned} \frac{\partial\phi_6(\theta;x)}{\partial\theta} &= e^{i\theta/2}A^{-\frac{1}{2}}\left[\frac{i}{2} - \frac{1}{2}A^{-1}\frac{\partial}{\partial\theta}\left\{\left(1 - i\theta x - \frac{\theta^2x^2}{6} + \cdots\right)\right.\right. \\ &\qquad \left.\left. + i\theta\left(1 - \frac{i\theta x}{3} + \cdots\right)\right\}\right], \\ A &= \cos\sqrt{2i\theta x} + i\theta\,\frac{\sin\sqrt{2i\theta x}}{\sqrt{2i\theta x}}. \end{aligned}$$

Therefore $\partial\phi_6(\theta;x)/\partial\theta|_{\theta=0} = ix/2$ so that (6.43) holds.

4.4 Noting that $F_6(0) = P(S_6 \le 0) = P(w^2(1) \le 1)$, we obtain $F_6(0) = 1-2\Phi(-1) = 1 - 2 \times 0.15866 = 0.68268$.

4.5 We have only to show that

(S8) $$\mathrm{Im}\left[\cos(-\theta^2)^{\frac{1}{4}}\cosh(-\theta^2)^{\frac{1}{4}}\right] = 0\,.$$

Since $-\theta^2 = \theta^2 \exp\{i(2n+1)\pi\}$ for $n = 0, \pm 1, \pm 2, \cdots$, we have

$$\begin{aligned}
\cos(-\theta^2)^{\frac{1}{4}} &= \cos(x+iy) \\
&= \cos x \cosh y - i \sin x \sinh y, \\
\cosh(-\theta^2)^{\frac{1}{4}} &= \cosh(x+iy) \\
&= \cosh x \cos y + i \sinh x \sin y,
\end{aligned}$$

where $x = \sqrt{\theta}\cos\{(2n+1)\pi/4\}$ and $y = \sqrt{\theta}\sin\{(2n+1)\pi/4\}$. Thus $y = x$ or $y = -x$ and it can be easily checked that (S8) holds.

4.6 We show that $F_7(-x) = 1 - F_7(x)$ for any x. Since $g_7(\theta; -x) = g_7(-\theta; x)$, we have

$$\begin{aligned}
F_7(-x) &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{1}{\theta}\,\mathrm{Im}[\phi_7(-\theta; x)]d\theta \\
&= \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{1}{\theta}\,\mathrm{Im}[\phi_7(\theta; x)]d\theta \\
&= 1 - F_7(x).
\end{aligned}$$

Chapter 7.

1.1 Putting $x_j = 1$ in (7.2) with $u_j = \varepsilon_j$ we have $y_j = \rho y_{j-1} + (1-\rho)\beta + \varepsilon_j$. Thus $\alpha = (1-\rho)\beta$ must be 0 when $\rho = 1$. Putting $x_j = (1, j)'$ and $\beta = (\beta_1, \beta_2)'$ in (7.2) with $u_j = \varepsilon_j$ leads us to $y_j = \rho y_{j-1} + \beta_1(1-\rho) + \beta_2\rho + j\beta_2(1-\rho) + \varepsilon_j$ so that $\gamma = (1-\rho)\beta_2$ must be 0 when $\rho = 1$.

1.2 The LSE $\hat{\rho}$ is given by

$$\hat{\rho} = \frac{\sum_{j=2}^{T}(y_{j-1} - \bar{y}_{-1})(y_j - \bar{y}_0)}{\sum_{j=2}^{T}(y_{j-1} - \bar{y}_{-1})^2} = \frac{\sum_{j=2}^{T}(y_{j-1} - \bar{y}_{-1})\varepsilon_j}{\sum_{j=2}^{T} y_{j-1}^2 - (T-1)\bar{y}_{-1}^2} + 1\,,$$

where $\bar{y}_{-1} = \sum_{j=2}^{T} y_{j-1}/(T-1)$ and $\bar{y}_0 = \sum_{j=2}^{T} y_j/(T-1)$. Since $y_j = \alpha j + \varepsilon_1 + \cdots + \varepsilon_j$, we have

$$\operatorname*{plim}_{T\to\infty} = \frac{1}{T^3}\sum_{j=2}^{T} y_{j-1}^2 = \frac{\alpha^2}{3}, \qquad \operatorname*{plim}_{T\to\infty} = \frac{1}{T^2}\bar{y}_{-1}^2 = \frac{\alpha^2}{4}\,,$$

$$\begin{aligned}\frac{1}{T\sqrt{T}}\sum_{j=2}^{T}(y_{j-1}-\bar{y}_{-1})\varepsilon_j &= \frac{\alpha}{T\sqrt{T}}\sum_{j=2}^{T}\left(j-1-\frac{T}{2}\right)\varepsilon_j + o_p(1)\\ &\longrightarrow N\left(0,\frac{\alpha^2\sigma^2}{12}\right).\end{aligned}$$

Thus we obtain $T\sqrt{T}(\hat{\rho}-1) \to N(0, 12\sigma^2/\alpha^2)$.

1.3 The first equality is obvious. Consider

$$\begin{aligned}2\sum_{j=2}^{T}\hat{\eta}_{j-1}(\hat{\eta}_j-\hat{\eta}_{j-1}) &= -\sum_{j=2}^{T}(\hat{\eta}_j-\hat{\eta}_{j-1})^2+\sum_{j=2}^{T}\hat{\eta}_j^2-\sum_{j=2}^{T}\hat{\eta}_{j-1}^2\\ &= -\sum_{j=2}^{T}(\hat{\eta}_j-\hat{\eta}_{j-1})^2+\hat{\eta}_T^2-\hat{\eta}_1^2,\end{aligned}$$

which yields U_T in (7.12).

1.4 Consider first

$$\frac{1}{T}E\left(\sum_{j=1}^{T}u_j\right)^2=\frac{1}{T}\sum_{j,k=1}^{T}\gamma_{j-k}=\sum_{j=-(T-1)}^{T-1}\left(1-\frac{|\,j\,|}{T}\right)\gamma_j\,,$$

which converges to $\displaystyle\sum_{j=-\infty}^{\infty}\gamma_j=2\pi f(0)=\sigma^2\left(\sum_{l=0}^{\infty}\alpha_l\right)^2=\sigma_L^2$. Since $E(u_j^2)=\gamma_0=\sigma^2\displaystyle\sum_{l=0}^{\infty}\alpha_l^2$, we have $\sigma_S^2=\gamma_0$.

2.1 It is easy to see that

$$\begin{aligned}\frac{1}{T}\sum_{j=2}^{T}(\eta_j-\eta_{j-1})^2 &= \frac{1}{T}\sum_{j=2}^{T}\left(-\frac{c}{T}\eta_{j-1}+u_j\right)^2\\ &= \frac{1}{T}\sum_{j=2}^{T}u_j^2+o_p(1)\longrightarrow\sigma_S^2\quad\text{in probability.}\end{aligned}$$

We also have that

$$\begin{aligned}\frac{\eta_1^2}{T}-\sigma_L^2Z^2 &= \frac{1}{T}\left(\sqrt{T}\rho\sigma_LZ+u_1\right)^2-\sigma_L^2Z^2\\ &= (\rho^2-1)\sigma_L^2Z^2+\frac{2}{\sqrt{T}}\rho\sigma_LZu_1+\frac{1}{T}u_1^2,\end{aligned}$$

which evidently converges in probability to 0.

2.2 Defining a continuous function on C :

$$h(x) = \left(\frac{1}{2}(x^2(1) - x^2(0) - \sigma_S^2),\ \int_0^1 x^2(t)dt\right),$$

we can deduce that $h(Y_T) - (U_{1T}, V_{1T}) \to 0$ in probability. Since $\mathcal{L}(Y_T) \to \mathcal{L}(\sigma_L Y)$, we can establish (7.29) by the continuous mapping theorem.

2.3 It is easy to see that

$$\mathcal{L}\left(\gamma\left(\frac{U_1}{V_1} + c\right)\right) \longrightarrow \mathcal{L}\left(\frac{\displaystyle\int_0^1 e^{-ct}dw(t)}{\displaystyle\int_0^1 e^{-2ct}dt}\right),$$

where

$$\int_0^1 e^{-2ct}dt = \frac{\sinh c}{ce^c},\quad \int_0^1 e^{-ct}dw(t) \sim N\left(0, \frac{\sinh c}{ce^c}\right),$$

which establishes (7.34).

2.4 Since $\hat{\eta}_j - \hat{\eta}_{j-1} = \eta_j - \eta_{j-1}$, (7.39) holds because of (7.27). Noting that $\hat{\eta}_1 = \eta_1 - \bar{\eta} = \sqrt{T}\rho\sigma_L Y(0) + u_1 - \bar{\eta}$ and $\bar{\eta} = \sum_{j=1}^{T} Y_T(j/T)/\sqrt{T}$, we can also establish (7.40).

2.5 It follows from (7.23) and (7.24) that

$$\begin{aligned}
\gamma\left(\frac{U_2}{V_2} + c\right) &= \frac{\gamma\left\{\displaystyle\int_0^1 Y(t)dw(t) - w(1)\int_0^1 Y(t)dt + \frac{1}{2}(1-r)\right\}}{\displaystyle\int_0^1 Y^2(t)dt - \left(\int_0^1 Y(t)dt\right)^2} \\
&= \frac{\displaystyle\int_0^1 e^{-ct}dw(t) - w(1)\int_0^1 e^{-ct}dt}{\displaystyle\int_0^1 e^{-2ct}dt - \left(\int_0^1 e^{-ct}dt\right)^2} + o_p(1),
\end{aligned}$$

where

$$\int_0^1 e^{-ct}dw(t) - w(1)\int_0^1 e^{-ct}dt \sim N\left(0, \int_0^1 e^{-2ct}dt - \left(\int_0^1 e^{-ct}dt\right)^2\right),$$

$$\int_0^1 e^{-2ct}dt - \left(\int_0^1 e^{-ct}dt\right)^2 = \frac{c\sinh c - 2\cosh c + 2}{c^2e^c}$$

This leads us to the conclusion.

2.6 Fuller's estimator $\tilde{\rho}$ gives

$$T(\tilde{\rho}-1)=\frac{\dfrac{1}{T}\displaystyle\sum_{j=2}^{T}(y_{j-1}-\bar{y}_{-1})y_j}{\dfrac{1}{T^2}\displaystyle\sum_{j=2}^{T}(y_{j-1}-\bar{y}_{-1})^2}=\frac{\dfrac{1}{T}\displaystyle\sum_{j=2}^{T}(\eta_{j-1}-\bar{\eta}_{-1})\eta_j}{\dfrac{1}{T^2}\displaystyle\sum_{j=2}^{T}(\eta_{j-1}-\bar{\eta}_{-1})^2},$$

where $\bar{y}_{-1}=\sum_{j=2}^{T}y_{j-1}/(T-1)$ and $\bar{\eta}_{-1}=\sum_{j=2}^{T}\eta_{j-1}/(T-1)$. Since

$$T(\hat{\rho}-1)=\frac{\dfrac{1}{T}\displaystyle\sum_{j=2}^{T}(\eta_{j-1}-\bar{\eta})(\eta_j-\bar{\eta})}{\dfrac{1}{T^2}\displaystyle\sum_{j=2}^{T}(\eta_{j-1}-\bar{\eta})^2},$$

it can be checked easily that the limiting distribution of $T(\tilde{\rho}-1)$ is the same as that of $T(\hat{\rho}-1)$ given in (7.41).

2.7 It holds that

$$\hat{\eta}_j-\hat{\eta}_{j-1}=\eta_j-\eta_{j-1}-\left(\frac{3}{T^3}+O\left(\frac{1}{T^4}\right)\right)\sum_{k=1}^{T}k\eta_k,$$

where $\sum_{k=1}^{T}k\eta_k=O_p\left(T^2\sqrt{T}\right)$. Then it is seen that (7.49) holds because of (7.27). Since

$$\begin{aligned}\frac{\hat{\eta}_1^2}{T} &= \frac{1}{T}\left(\sqrt{T}\rho\sigma_L Y(0)+u_1-\sum_{k=1}^{T}k\eta_k\Big/\sum_{k=1}^{T}k^2\right)^2\\ &= \sigma_L^2Y^2(0)+o_p(1),\end{aligned}$$

(7.50) can also be established.

2.8 Since $c=0$ and $Y(0)=\gamma$, we have $Y(t)=\gamma+w(t)$ so that, as $|\,\gamma\,|\to\infty$,

$$\begin{aligned}\frac{U_3}{\gamma^2} &= \frac{1}{2}\left(\left(1-3\int_0^1 tdt\right)^2-1\right)+o_p(1)=-\frac{3}{8}+o_p(1),\\ \frac{V_3}{\gamma^2} &= 1-3\left(\int_0^1 tdt\right)^2+o_p(1)=\frac{1}{4}+o_p(1).\end{aligned}$$

Thus U_3/V_3 converges in probability to $-3/2$ as $|\,\gamma\,| \to \infty$.

2.9 The LSE $\tilde{\rho}$ in the present case gives

$$\begin{aligned} T(\tilde{\rho}-1) &= \frac{1}{T}\sum_{j=2}^{T}\left(y_{j-1} - \frac{j\sum_{k=2}^{T} k y_{k-1}}{\sum_{k=2}^{T} k^2}\right) \\ &\qquad \times (y_j - y_{j-1}) \Bigg/ \left[\frac{1}{T^2}\sum_{j=2}^{T}\left(y_{j-1} - \frac{j\sum_{k=2}^{T} k y_{k-1}}{\sum_{k=2}^{T} k^2}\right)^2\right], \end{aligned}$$

where $y_j = \rho y_{j-1} + \varepsilon_j$. Then the FCLT and the continuous mapping theorem yield

$$\mathcal{L}(T(\tilde{\rho}-1)) \longrightarrow \mathcal{L}\left(\frac{\int_0^1 Y(t)dY(t) - 3\int_0^1 tY(t)dt\int_0^1 tdY(t)}{\int_0^1\left(Y(t) - 3t\int_0^1 sY(s)ds\right)^2 dt}\right),$$

where $dY(t) = -cY(t)dt + dw(t)$.

2.10 It can be checked that

$$\hat{\eta}_j - \hat{\eta}_{j-1} = \eta_j - \eta_{j-1} + \left(\frac{12}{T^4} + O\left(\frac{1}{T^5}\right)\right)\left(\frac{T^2}{2}\sum_{k=1}^{T}\eta_k - T\sum_{k=1}^{T} k\eta_k\right),$$

where $\sum_{k=1}^{T}\eta_k = O_p\left(T\sqrt{T}\right)$ and $\sum_{k=1}^{T} k\eta_k = O_p\left(T^2\sqrt{T}\right)$. Thus we obtain (7.55) because of (7.27). We also have

$$\begin{aligned} \hat{\eta}_1 &= \eta_1 - \frac{12}{T^4}\left[\left(\frac{T^3}{3} - \frac{T^2}{2}\right)\sum_{k=1}^{T}\eta_k + \left(T - \frac{T^2}{2}\right)\sum_{k=1}^{T} k\eta_k\right] + o_p(\sqrt{T}) \\ &= \sqrt{T}\sigma_L Y(0) - \frac{4}{T}\sum_{k=1}^{T}\eta_k + \frac{6}{T^2}\sum_{k=1}^{T} k\eta_k + o_p(\sqrt{T}) \end{aligned}$$

so that (7.56) holds.

2.11 The normal equations for the LSE's $\tilde{a}, \tilde{b}$ and $-\tilde{c}$ in the model (7.59) are :

$$\begin{pmatrix} 1 & \frac{1}{2} & \int_0^1 \tilde{Y}(t)dt \\ \frac{1}{2} & \frac{1}{3} & \int_0^1 t\tilde{Y}(t)dt \\ \int_0^1 \tilde{Y}(t)dt & \int_0^1 t\tilde{Y}(t)dt & \int_0^1 \tilde{Y}^2(t)dt \end{pmatrix} \begin{pmatrix} \tilde{a} \\ \tilde{b} \\ -\tilde{c} \end{pmatrix} = \begin{pmatrix} \int_0^1 d\tilde{Y}(t) \\ \int_0^1 t d\tilde{Y}(t) \\ \int_0^1 \tilde{Y}(t)d\tilde{Y}(t) \end{pmatrix}.$$

Solving for $-\tilde{c}$ we obtain $-\tilde{c} = U_4/V_4$ with Y and r replaced by $\tilde{Y}$ and 1, respectively, in the definitions of U_4 and V_4.

3.1 We have only to show that W_T can be expressed as in (7.62). Since

$$\begin{aligned} W_T &= -\frac{1}{T}\sum_{j=2}^{T} \hat{\eta}_{j-1}(\hat{\eta}_j + \hat{\eta}_{j-1}) \\ &= -\frac{1}{2T}\left[\sum_{j=2}^{T}(\hat{\eta}_j + \hat{\eta}_{j-1})^2 - \sum_{j=2}^{T}\hat{\eta}_j^2 + \sum_{j=2}^{T}\hat{\eta}_{j-1}^2\right], \end{aligned}$$

this last expression yields (7.62).

3.2 Suppose that T is even so that $N = T/2$ is an integer. Then we have, using (7.65),

$$\bar{\eta} = \frac{1}{T}\sum_{j=1}^{N}(\xi_{2j} - \xi_{2j-1}) = \frac{1}{T}\sum_{j=1}^{N}\left(-\frac{c}{T}\xi_{2j-1} + v_{2j}\right),$$

which is clearly $O_p(1/\sqrt{T})$. The case of T odd can also be proved similarly.

3.3 In the present case $-T(\hat{\rho} + 1) = W_T/X_T$, where

$$\begin{aligned} W_T &= \frac{1}{2T}\left[(\eta_T - \bar{\eta})^2 - (\eta_1 - \bar{\eta})^2 - \sum_{j=2}^{T}(\eta_j + \eta_{j-1} - 2\bar{\eta})^2\right] \\ &= \frac{1}{2T}\left[\eta_T^2 - \eta_1^2 - \sum_{j=2}^{T}(\eta_j + \eta_{j-1})^2\right] + o_p(1), \\ X_T &= \frac{1}{T^2}\left[\sum_{j=1}^{T}(\eta_j - \bar{\eta})^2 - (\eta_T - \bar{\eta})^2\right] \\ &= \frac{1}{T^2}\sum_{j=1}^{T}\eta_j^2 + o_p(1). \end{aligned}$$

Here we have used the fact that $\bar{\eta} = O_p(1/\sqrt{T})$. Then we can deduce (7.67) as in Model A.

3.4 Suppose that T is even so that $N = T/2$ is an integer. Then (7.65) leads us to

$$\begin{aligned}\sum_{k=1}^{T} k\eta_k &= \sum_{j=1}^{N}(2j\xi_{2j} - (2j-1)\xi_{2j-1}) \\ &= -\frac{2c}{T}\sum_{j=1}^{N} j\xi_{2j-1} + 2\sum_{j=1}^{N} jv_{2j} + \sum_{j=1}^{N}\xi_{2j-1}\,,\end{aligned}$$

which is $O_p(T\sqrt{T})$. The case of T odd can be dealt with similarly.

3.5 We have $-T(\hat{\rho}+1) = W_T/X_T$, where

$$\begin{aligned}W_T &= \frac{1}{2T}\left[\left(\eta_T - \frac{T\sum_{k=1}^{T} k\eta_k}{\sum_{k=1}^{T} k^2}\right)^2 - \left(\eta_1 - \frac{\sum_{k=1}^{T} k\eta_k}{\sum_{k=1}^{T} k^2}\right)^2\right. \\ &\qquad\left. - \sum_{j=2}^{T}\left(\eta_j + \eta_{j-1} - \frac{(2j-1)\sum_{k=1}^{T} k\eta_k}{\sum_{k=1}^{T} k^2}\right)^2\right] \\ &= \frac{1}{2T}\left[\eta_T^2 - \eta_1^2 - \sum_{j=2}^{T}(\eta_j + \eta_{j-1})^2\right] + o_p(1)\,, \\ X_T &= \frac{1}{T^2}\sum_{j=1}^{T}\left(\eta_j - \frac{j\sum_{k=1}^{T} k\eta_k}{\sum_{k=1}^{T} k^2}\right)^2 + o_p(1) \\ &= \frac{1}{T^2}\sum_{j=1}^{T}\eta_j^2 + o_p(1)\,.\end{aligned}$$

Here use has been made of the facts that $\sum_{k=1}^{T} k^2 = T^3/3 + O(T^2)$ and $\sum_{k=1}^{T} k\eta_k = O_p(T\sqrt{T})$. We can now deduce (7.67) as in Model A.

3.6 The present case is a mixture of Model B and Model C. We can prove (7.67) using the facts that $\bar{\eta} = O_p(1/\sqrt{T})$, $\sum_{k=1}^{T} k^2 = T^3/3 + O(T^2)$ and $\sum_{k=1}^{T} k\eta_k = O_p(T\sqrt{T})$.

4.1 Recalling the definitions of U_2 and V_2 in Theorem 7.2 and using (7.73), we have

$$m_{21}(\theta_1, \theta_2) = E\left[\exp\{\theta_1 U(Z) + \theta_2 V(Z)\}\frac{d\mu_Y}{d\mu_Z}(Z)\right],$$

where

$$\begin{aligned} U(Z) &= \frac{1}{2}(Z^2(1) - \gamma^2 - 1) - (Z(1) - \gamma)\int_0^1 Z(t)dt + \frac{1}{2}(1-r)\,, \\ V(Z) &= \int_0^1 Z^2(t)dt - \left(\int_0^1 Z(t)dt\right)^2, \\ \frac{d\mu_Y}{d\mu_Z}(Z) &= \exp\left\{\frac{\beta - c}{2}(Z^2(1) - \gamma^2 - 1) + \frac{\beta^2 - c^2}{2}\int_0^1 Z^2(t)dt\right\}. \end{aligned}$$

Putting $\beta = \sqrt{c^2 - 2\theta_2}$ yields the last expression in (7.75).

4.2 Noting that $W_l \sim N(\gamma\kappa_l, \Omega_l)$ we consider

$$\begin{aligned} & (w - \gamma\kappa_l)'\Omega_l^{-1}(w - \gamma\kappa_l) - w'A_l w - 2\gamma h_l' w \\ = \;& (w - \gamma g)'(\Omega_l^{-1} - A_l)(w - \gamma g) - \gamma^2 g'(\Omega_l^{-1} - A_l)g + \gamma^2\kappa_l'\Omega_l^{-1}\kappa_l\,, \end{aligned}$$

where $g = (\Omega_l^{-1} - A_l)^{-1}(\Omega_l^{-1}\kappa_l + h_l)$. Then (7.75) leads us to the first equality in (7.76). The second equality can be obtained by substituting $a = \beta + \theta_1 - c$ and

$$\begin{aligned} g'(\Omega_l^{-1} - A_l)g &= (\Omega_l^{-1}\kappa_l + h_l)'(\Omega_l^{-1} - A_l)^{-1}(\Omega_l^{-1}\kappa_l + h_l) \\ &= (\kappa_l + \Omega_l h_l)'(B_l - A_l\Omega_l)^{-1}\Omega_l^{-1}(\kappa_l + \Omega_l h_l) \\ &= (\kappa_l + \Omega_l h_l)'(\Omega_l^{-1} + A_l(B_l - \Omega_l A_l)^{-1})(\kappa_l + \Omega_l h_l) \\ &= \kappa_l'\Omega_l^{-1}\kappa_l + 2h_l'\kappa_l + h_l'\Omega_l h_l \\ &\quad\; + (\kappa_l + \Omega_l h_l)'A_l(B_l - \Omega_l A_l)^{-1}(\kappa_l + \Omega_l h_l)\,, \end{aligned}$$

where we have used the matrix identity $(B_l - A_l\Omega_l)^{-1} = B_l + A_l(B_l - \Omega_l A_l)^{-1}\Omega_l$.

4.3 Substituting $h_1 = 0$ and $A_1 = a$ into the last expression in (7.76), we have

$$m_{11}(\theta_1, \theta_2) = \exp\left[\frac{1}{2}\left\{c - r\theta_1 + \gamma^2\left(-a + \frac{a\kappa_1^2}{1 - a\Omega_1}\right)\right\}\right]\left[e^{\beta}(1 - a\Omega_1)\right]^{-\frac{1}{2}},$$

where $a = \beta + \theta_1 - c,\ \kappa_1 = e^{-\beta}$ and

$$\begin{aligned}
1 - a\Omega_1 &= 1 - (\beta + \theta_1 - c)\frac{1 - e^{-2\beta}}{2\beta} \\
&= e^{-\beta}\left[\cosh\beta + (c - \theta_1)\frac{\sinh\beta}{\beta}\right].
\end{aligned}$$

Then we can arrive at (7.77) easily.

4.4 The first equality is obvious, while the second equality comes from (7.75). This leads us to the last equality in (7.76) with γ^2 replaced by $Z^2(0)$. Then, noting that $E\left[\exp\{bZ^2(0)\}\right] = (1 - b/c)^{-\frac{1}{2}}$, we have the second last equality, which has the interpretation as given on the right side of the last equality.

5.1 For the case of $l = 1$ we consider the limiting distribution of

$$\frac{1}{\sqrt{2c}}\left(\frac{U_1}{V_1} + cr\right) = \frac{\sqrt{\frac{c}{2}}(U_1 + crV_1)}{cV_1}.$$

For this purpose we have, from (7.82),

$$\begin{aligned}
& m_{12}\left(\sqrt{\frac{c}{2}}\theta_1,\ \sqrt{\frac{c}{2}}cr\theta_1 + c\theta_2\right) \\
&= \exp\left\{\frac{1}{2}\left(c - \sqrt{\frac{c}{2}}r\theta_1\right)\right\}\left[\cosh\nu + \frac{2c^2 - \frac{c}{2}\theta_1^2 - 2\sqrt{\frac{c}{2}}cr\theta_1 - 2c\theta_2}{2c}\frac{\sinh\nu}{\nu}\right]^{-\frac{1}{2}},
\end{aligned}$$

where

$$\begin{aligned}
\nu &= \left(c^2 - 2\sqrt{\frac{c}{2}}cr\theta_1 - 2c\theta_2\right)^{\frac{1}{2}} \\
&= c - \sqrt{\frac{c}{2}}r\theta_1 - \theta_2 - \frac{r^2\theta_1^2}{4} + O\left(\frac{1}{\sqrt{c}}\right).
\end{aligned}$$

Then we can deduce that

$$m_{12}\left(\sqrt{\frac{c}{2}}\theta_1,\ \sqrt{\frac{c}{2}}cr\theta_1 + c\theta_2\right) \longrightarrow \exp\left[\frac{\theta_1^2}{2}\frac{r^2}{4} + \frac{\theta_2}{2}\right],$$

which implies that the limiting distribution is $\mathrm{N}(0, r^2)$.

5.2 Consider

$$m_{11}(-4ce^{2c}\theta_1\,,4c^2e^{2c}\theta_2\,|\,\gamma=0)=\exp\left\{\frac{1}{2}\left(c+4cre^{2c}\theta_1\right)\right\}H_1^{-\frac{1}{2}}\,,$$

where

$$H_1=\cosh\nu+\left(c+4ce^{2c}\theta_1\right)\frac{\sinh\nu}{\nu}\,,\quad \nu=\sqrt{c^2-8c^2e^{2c}\theta_2}\,.$$

Since

$$\frac{1}{\nu}=\left(c^2-8c^2e^{2c}\theta_2\right)^{-\frac{1}{2}}=-\frac{1}{c}\left(1+4e^{2c}\theta_2+O\left(e^{4c}\right)\right)\,,$$

we obtain $H_1=(1-2\theta_1-2\theta_2)\,e^c+O\left(e^{3c}\right)$. Thus it holds that $m_{11}\longrightarrow(1-2\theta_1-2\theta_2)^{-\frac{1}{2}}$, which leads us to the conclusion.

5.3 Let us consider

$$m_{41}(ce^c\theta_1,c^2e^c\theta_1+2c^2e^{2c}\theta_2\,|\,\gamma=0)=\exp\left\{\frac{c-rce^c\theta_1}{2}\right\}A^{-\frac{1}{2}}\,,$$

where

$$\begin{aligned}
A &= \left(\frac{A_1}{\nu^5}-\frac{A_2}{\nu^7}\right)\sinh\nu+\left(\frac{c^4}{\nu^4}+\frac{A_3}{\nu^6}+\frac{A_2}{\nu^8}\right)\cosh\nu+\frac{A_4}{\nu^6}-\frac{A_2}{\nu^8}\,,\\
\nu &= (c^2-2c^2e^c\theta_1-4c^2e^{2c}\theta_2)^{\frac{1}{2}}=-c+O(ce^c),\\
\frac{c^4}{\nu^4} &= 1+4e^c\theta_1+8e^{2c}\theta_2+12e^{2c}\theta_1^2+O(e^{3c}),\\
\frac{c^5}{\nu^5} &= -(1+5e^c\theta_1+10e^{2c}\theta_2+\frac{35}{2}e^{2c}\theta_1^2+O(e^{3c})),\\
\frac{c^6}{\nu^6} &= 1+O(e^c),\qquad \frac{c^7}{\nu^7}=-1+O(e^c),\qquad \frac{c^8}{\nu^8}=1+O(e^c),\\
A_1 &= c^5-c^5e^c\theta_1-8c^2(c^2-3c-3)e^c\theta_1+O(c^4e^{2c}),\\
A_2 &= 24c^5e^c\theta_1+O(c^5e^{2c}),\\
A_3 &= -8c^5e^c\theta_1+O(c^5e^{2c}),\qquad A_4=O(c^5e^c).
\end{aligned}$$

We then have

$$A = \frac{1}{2}\left[\left\{\frac{-1}{c^5}\left(1+5e^c\theta_1+10e^{2c}\theta_2+\frac{35}{2}e^{2c}\theta_1^2+O(e^{3c})\right)\right.\right.$$

$$
\begin{aligned}
& \times \left(c^5 - c^5 e^c \theta_1 - 8c^2(c^2 - 3c - 3) e^c \theta_1 + O(c^4 e^{2c})\right) \\
& + \frac{1}{c^7}(1 + O(e^c)) \left(24 c^5 e^c \theta_1 + O(c^5 e^{2c})\right)\Big\} (e^\nu - e^{-\nu}) \\
& + \Big\{1 + 4e^c\theta_1 + 8e^{2c}\theta_2 + 12 e^{2c}\theta_1^2 + O(e^{3c}) \\
& \qquad + \frac{1}{c^6}(1 + O(e^c)) \left(-8c^5 e^c \theta_1 + O(c^5 e^{2c})\right) \\
& + \frac{1}{c^8}(1 + O(e^c)) \left(24 c^5 e^c \theta_1 + O(c^5 e^{2c})\right)\Big\} (e^\nu + e^{-\nu})\Big] + O\left(\frac{e^c}{c}\right) \\
= \;& (1 - \theta_2 - \frac{1}{4}\theta_1^2) e^c + O\left(\frac{e^c}{c}\right)
\end{aligned}
$$

so that $m_{41} \to (1 - \theta_2 - \theta_1^2/4)^{-\frac{1}{2}}$. This last m.g.f. is that of $(XY/2, X^2/2)$, where $(X, Y)' \sim \mathrm{N}(0, I_2)$. Thus we obtain the required result.

5.4 Consider

$$
m_{31}\left(c^3 e^{2c}\theta\,,\, c^4 e^{2c}\theta \,|\, \gamma = 0\right) = \exp\left\{\frac{1}{2}\left(c - r c^3 e^{2c}\theta\right)\right\} H_3^{-\frac{1}{2}}\,,
$$

where

$$
\begin{aligned}
H_3 &= \frac{c^3}{\nu^3}\left\{1 - e^{2c}\theta\left(c^2 + 3c + 3\right)\right\}\sinh\nu + \frac{c^2}{\nu^2}\cosh\nu \\
&\quad -3c^3 e^{2c}\theta\left\{c^2 + 3c + 3 - 2c\,(c+1)\right\}\left(\frac{\sinh\nu}{\nu^5} - \frac{\cosh\nu}{\nu^4}\right)\,, \\
\nu &= \sqrt{c^2\left(1 - 2c^2 e^{2c}\theta\right)}\,.
\end{aligned}
$$

Since we have

$$
\frac{1}{\nu^k} = \frac{(-1)^k}{c^k}\left(1 + k c^2 e^{2c}\theta + O\left(c^4 e^{4c}\right)\right)\,, \quad (k = 1, 2, \cdots)\,,
$$

we obtain

$$
H_3 = e^c\left\{1 + \frac{3}{2}\theta + O\left(\frac{1}{c}\right)\right\}\,.
$$

Thus $m_{31} \longrightarrow (1 + 3\theta/2)^{-\frac{1}{2}}$, which is the required result.

5.5 Consider

$$
m_{31}\left(c^2 e^{2c}\theta\,, c^3 e^{2c}\theta\right) = \exp\left\{\frac{1}{2}\left(c - r c^2 e^{2c}\theta + \frac{G\gamma^2}{H_3}\right)\right\} H_3^{-\frac{1}{2}}\,,
$$

where

$$
\begin{aligned}
G &= e^{2c}\theta\left[\left\{2c^5-\left(c^2+3c+3\right)\left(c^4e^{2c}\theta-3c^2+2c^3\right)+3c^4\right\}\left(-\frac{\sinh\nu}{\nu^3}\right)\right.\\
&\quad -3\left\{2c^5-c^6e^{2c}\theta+3c^4-2c^5-3c^5e^{2c}\theta-3c^4e^{2c}\theta\right\}\frac{\sinh\nu}{\nu^5}\\
&\quad +3\left\{2c^5-c^6e^{2c}\theta+5c^4-2c^5-(c+1)\left(3c^4e^{2c}\theta-6c^2+4c^3\right)\right\}\frac{\cosh\nu}{\nu^5}\\
&\quad \left.-\frac{6}{\nu^4}\left\{c^4+(c+1)\left(3c^2-2c^3\right)\right\}\right],\\
H_3 &= \left\{c^2e^{2c}\theta\left(c^2+3c+3\right)-c^3\right\}\left(-\frac{\sinh\nu}{\nu^3}\right)+\frac{c^2}{\nu^2}\cosh\nu\\
&\quad -3c^2e^{2c}\theta\left\{c^2+3c+3-2c(c+1)\right\}\left(\frac{\sinh\nu}{\nu^5}-\frac{\cosh\nu}{\nu^4}\right),\\
\frac{1}{\nu^k} &= \left(c^2-2c^3e^{2c}\theta\right)^{-\frac{k}{2}}\\
&= \frac{(-1)^k}{c^k}\left(1+kce^{2c}\theta+O\left(c^2e^{4c}\right)\right), \quad (k=1,2,\cdots).
\end{aligned}
$$

Then we obtain

$$
G=e^c\left\{3\theta+O\left(\frac{1}{c}\right)\right\}, \quad H_3=e^c\left\{1+O\left(\frac{1}{c}\right)\right\},
$$

so that $m_{31}\longrightarrow\exp\left(3\gamma^2\theta/2\right)$.

5.6 Let us consider

$$
\begin{aligned}
&m_{21}(\sqrt{-c}\,e^c\theta_1,\, c\sqrt{-c}\,e^c\theta_1+ce^{2c}\theta_2)\\
&\quad =\exp\left[\frac{c-r\sqrt{-c}\,e^c\theta_1}{2}-\frac{\gamma^2c^3e^{2c}(\theta_1^2-2\theta_2)}{2H_2}A\right]H_2^{-\frac{1}{2}},
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \frac{\sinh\nu}{\nu^3}-\frac{2}{\nu^4}(\cosh\nu-1),\\
\nu &= (c^2-2c\sqrt{-c}\,e^c\theta_1-2ce^{2c}\theta_2)^{\frac{1}{2}}=-c+O(\sqrt{-c}\,e^c),\\
\frac{c}{\nu} &= -1+\frac{e^c}{\sqrt{-c}}\theta_1-\frac{e^{2c}}{c}\theta_2+\frac{3e^{2c}}{2c}\theta_1^2+O\left(\frac{e^{3c}}{c\sqrt{-c}}\right),\\
H_2 &= \frac{-ce^{2c}\theta_1^2+c^2\sqrt{-c}\,e^c\theta_1-c^3+2c\sqrt{-c}\,e^c\theta_1+2ce^{2c}\theta_2}{-\nu^2}\frac{\sinh\nu}{\nu}+\frac{c^2}{\nu^2}\cosh\nu\\
&\quad +\frac{2(-ce^{2c}\theta_1^2+c^2\sqrt{-c}\,e^c\theta_1-2c^2\sqrt{-c}\,e^c\theta_1-2c^2e^{2c}\theta_2)}{\nu^4}(\cosh\nu-1).
\end{aligned}
$$

Since

$$\begin{aligned}
\frac{c^2}{\nu^2} &= 1 - \frac{2e^c}{\sqrt{-c}}\theta_1 + O\left(\frac{e^{2c}}{c}\right), \\
\frac{c^3}{\nu^3} &= -1 + \frac{3e^c}{\sqrt{-c}}\theta_1 + O\left(\frac{e^{2c}}{c}\right), \\
\frac{c^4}{\nu^4} &= 1 - \frac{4e^c}{\sqrt{-c}}\theta_1 + O\left(\frac{e^{2c}}{c}\right),
\end{aligned}$$

we obtain $H_2 = e^c + O(e^c/c)$ and

$$c^3 e^{2c} A = -\frac{1}{2}e^c + O\left(\frac{e^c}{c}\right).$$

Then we have $m_{21} \to \exp\{\gamma^2(\theta_1^2 - 2\theta_2)/4\}$, as in the case of $l = 1$.

6.1 We have only to identify the matrices A_l and B_l given below (7.92). Consider the case of $l = 1$. Putting $y = (y_0, y_1, \cdots, y_T)'$ with $y_0 = \varepsilon_0/\sqrt{1-\rho^2}$ and $\varepsilon = (\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_T)'$ we have

$$T(\tilde{\rho}_1 - \rho) = \frac{1}{T\sigma^2}\sum_{j=1}^{T} y_{j-1}\varepsilon_j \Bigg/ \left[\frac{1}{T^2\sigma^2}\sum_{j=1}^{T} y_{j-1}^2\right],$$

where

$$\begin{aligned}
\sum_{j=1}^{T} y_{j-1}\varepsilon_j &= \frac{1}{2}\left[y'\begin{pmatrix} \underset{\sim}{0} & I_T \\ 0 & \underset{\sim}{0}' \end{pmatrix}\varepsilon + \varepsilon'\begin{pmatrix} \underset{\sim}{0}' & 0 \\ I_T & \underset{\sim}{0} \end{pmatrix} y\right], \\
\sum_{j=1}^{T} y_{j-1}^2 &= y'\begin{pmatrix} I_T & \underset{\sim}{0} \\ \underset{\sim}{0}' & 0 \end{pmatrix} y\,.
\end{aligned}$$

Noting that

$$y = \begin{pmatrix} & C_{11} & & & \underset{\sim}{0} \\ \rho^T\Big/\sqrt{1-\rho^2} & \rho^{T-1} & \cdots & \rho & 1 \end{pmatrix}\varepsilon,$$

we obtain $P(T(\tilde{\rho}_1 - \rho) \le x) = P(\varepsilon'(xB_1 - A_1)\varepsilon/\sigma^2 \ge 0)$, which yields (7.92). The case of $l = 2$ can be proved similarly.

6.2 Noting that $1/\sqrt{1-\rho^2} = \sqrt{T}/\sqrt{2c} + O(1/\sqrt{T})$, we may put $y_0 = \varepsilon_0/\sqrt{1-\rho^2} = \sqrt{T}\sigma Z + R_T$, where $Z \sim \mathrm{N}(0, 1/(2c))$ and $R_T = O_p(1/\sqrt{T})$. Then it is evident that

$\mathcal{L}(T(\tilde{\rho}_l - 1)) \to \mathcal{L}(U_l/V_l)$, where U_l and V_l are defined in Section 2 with $Y(0) \sim$ N $(0, 1/(2c))$ and $r = 1$. Since $T(\tilde{\rho}_l - \rho) = T(\tilde{\rho}_l - 1) + c$, we have that $P(T(\tilde{\rho}_l - \rho) \leq x) \to P(zV_l - U_l \geq 0)$, which yields (7.95).

7.1 As for Model D we have

$$\underset{\sim}{y}_j = \underset{\sim}{\beta}_0 + j\,\underset{\sim}{\beta}_1 + \underset{\sim}{\eta}_j = ((1, j) \otimes I_m)\beta + \underset{\sim}{\eta}_j$$

so that $y = ((e, d) \otimes I_m)\beta + \eta$. Since

$$\underset{\sim}{\eta}_j = \rho_m\,\underset{\sim}{\eta}_{j-1} + \underset{\sim}{u}_j = \rho_m^{j-1}\,\underset{\sim}{u}_1 + \rho_m^{j-2}\,\underset{\sim}{u}_2 + \cdots + \rho_m\,\underset{\sim}{u}_{j-1} + \underset{\sim}{u}_j\,,$$

it is easy to see that $\eta = (C(\rho_m) \otimes I_m)\,u$.

7.2 Because of the definition of $\hat{\rho}_m$ we have

$$N(\hat{\rho}_m - 1) = \frac{1}{N}\sum_{j=2}^{N} \underset{\sim}{\hat{\eta}}'_{j-1}\left(\underset{\sim}{\hat{\eta}}_j - \underset{\sim}{\hat{\eta}}_{j-1}\right) \Big/ \left[\frac{1}{N^2}\sum_{j=2}^{N} \underset{\sim}{\hat{\eta}}'_{j-1}\underset{\sim}{\hat{\eta}}_{j-1}\right],$$

where

$$\begin{aligned}
&2\sum_{j=2}^{N} \underset{\sim}{\hat{\eta}}'_{j-1}\left(\underset{\sim}{\hat{\eta}}_j - \underset{\sim}{\hat{\eta}}_{j-1}\right) \\
&\quad = \; -\sum_{j=2}^{N}\left(\underset{\sim}{\hat{\eta}}_j - \underset{\sim}{\hat{\eta}}_{j-1}\right)'\left(\underset{\sim}{\hat{\eta}}_j - \underset{\sim}{\hat{\eta}}_{j-1}\right) + \sum_{j=2}^{N} \underset{\sim}{\hat{\eta}}'_j\underset{\sim}{\hat{\eta}}_j - \sum_{j=2}^{N} \underset{\sim}{\hat{\eta}}'_{j-1}\underset{\sim}{\hat{\eta}}_{j-1} \\
&\quad = \; \underset{\sim}{\hat{\eta}}'_N\,\underset{\sim}{\hat{\eta}}_N - \sum_{j=1}^{N}\left(\underset{\sim}{\hat{\eta}}_j - \underset{\sim}{\hat{\eta}}_{j-1}\right)'\left(\underset{\sim}{\hat{\eta}}_j - \underset{\sim}{\hat{\eta}}_{j-1}\right).
\end{aligned}$$

Since $\underset{\sim}{\hat{\eta}}_N = (\underset{\sim}{e}'_N \otimes I_m)\hat{\eta} = (\underset{\sim}{e}'_N \otimes I_m)(\bar{M}C(\rho_m) \otimes I_m)u$, we arrive at the last expression for U_N. The expressions for V_N can be verified easily.

7.3 Since $\hat{\beta} = \beta + (d'd)^{-1}(d' \otimes I_m)\eta$ and $\underset{\sim}{y}_j = \beta j + \underset{\sim}{\eta}_j$, we first have

$$\underset{\sim}{\hat{\eta}}_j = \underset{\sim}{y}_j - j\hat{\beta} = \underset{\sim}{\eta}_j - j(d'd)^{-1}(d' \otimes I_m)\eta = \underset{\sim}{\eta}_j - j(d'd)^{-1}\sum_{k=1}^{N} k\,\underset{\sim}{\eta}_k$$

so that

$$\underset{\sim}{\hat{\eta}}_j - \underset{\sim}{\hat{\eta}}_{j-1} = \underset{\sim}{\eta}_j - \underset{\sim}{\eta}_{j-1} - \left(\frac{3}{N^3} + O\left(\frac{1}{N^4}\right)\right)\sum_{k=1}^{N} k\,\underset{\sim}{\eta}_k\,,$$

where $\sum_{k=1}^{N} k \underset{\sim}{\eta}_k = O_p(T^2\sqrt{T})$. Then the weak law of large numbers ensures that

$$\begin{aligned}
\frac{1}{N}\sum_{j=1}^{N}(\hat{\underset{\sim}{\eta}}_j - \hat{\underset{\sim}{\eta}}_{j-1})'(\hat{\underset{\sim}{\eta}}_j - \hat{\underset{\sim}{\eta}}_{j-1}) &= \frac{1}{N}\sum_{j=1}^{N}(\underset{\sim}{\eta}_j - \underset{\sim}{\eta}_{j-1})'(\underset{\sim}{\eta}_j - \underset{\sim}{\eta}_{j-1}) + o_p(1) \\
&= \frac{1}{N}\sum_{j=1}^{N} \underset{\sim}{u}_j' \underset{\sim}{u}_j + o_p(1) \\
&\longrightarrow m\sigma^2 \sum_{l=0}^{\infty} \alpha_l^2
\end{aligned}$$

in probability.

7.4 In the present case we have $\bar{M} = I_N$ and

$$K_N = xK_{2N} - K_{1N} = \frac{x}{N}C'(\rho_m)C(\rho_m) - \frac{1}{2}C'(\rho_m)\,\underset{\sim}{e}_N\,\underset{\sim}{e}_N' C(\rho_m).$$

It can be shown after some algebra (Nabeya and Tanaka (1990a)) that

$$\lim_{N\to\infty}\max_{j,k}\left|K_N(j,k) - K\left(\frac{j}{N},\frac{k}{N}\right)\right| = 0\,,$$

where $K(s,t) = \left\{x(e^{-c|\,s-t\,|} - e^{-c(2-s-t)})\big/\,c - e^{-c(2-s-t)}\right\}/\,2$. Since

$$xV_N - U_N = \frac{1}{N}\sum_{j,k=1}^{N} K_N(j,k)\,\underset{\sim}{u}_j'\,\underset{\sim}{u}_k + \frac{m\sigma^2}{2}\sum_{l=0}^{\infty}\alpha_l^2 + o_p(1),$$

(7.107) follows from the arguments in Section 6 of Chapter 5.

7.5 Put $F(x) = P\left(\int_0^1 \underset{\sim}{w}'(t)\,\underset{\sim}{w}(t)dt \le x\right)$ and $f(x) = dF(x)\,/\,dx$. Then

$$\begin{aligned}
\int_0^\infty e^{-\theta x} f(x)dx &= \left(\cosh\sqrt{2\theta}\right)^{-\frac{m}{2}} \\
&= 2^{\frac{m}{2}} e^{-m\sqrt{\theta/2}}\left(1 + e^{-2\sqrt{2\theta}}\right)^{-\frac{m}{2}} \\
&= 2^{\frac{m}{2}} \sum_{k=0}^{\infty} \begin{pmatrix} -\dfrac{m}{2} \\ k \end{pmatrix} e^{-(2k+\frac{m}{2})\sqrt{2\theta}}
\end{aligned}$$

so that, taking the inverse Laplace transform,

$$f(x) \;=\; \sum_{k=0}^{\infty} \begin{pmatrix} -\dfrac{m}{2} \\ k \end{pmatrix} \frac{2^{\frac{m}{2}}}{2\sqrt{\pi x^3}} b_k e^{-b_k^2/(4x)}, \quad \left(b_k = \sqrt{2}\left(2k + \frac{m}{2}\right)\right),$$

$$
\begin{aligned}
F(x) &= \int_0^x f(a)da \\
&= \sum_{k=0}^{\infty} \begin{pmatrix} -\frac{m}{2} \\ k \end{pmatrix} \frac{2^{(m+1)/2}}{\sqrt{\pi}} \int_{b_k/\sqrt{2x}}^{\infty} e^{-a^2/2} da \\
&= 2^{(m+2)/2} \sum_{k=0}^{\infty} \begin{pmatrix} -\frac{m}{2} \\ k \end{pmatrix} \Phi\left(-\frac{2k+\frac{m}{2}}{\sqrt{x}} \right).
\end{aligned}
$$

7.6 Noting that $\underset{\sim}{y}_0 = 0$ and $\underset{\sim}{y}_j = \underset{\sim}{y}_{j-1} + \underset{\sim}{\varepsilon}_j = \underset{\sim}{\varepsilon}_1 + \cdots + \underset{\sim}{\varepsilon}_j$ we obtain

$$
\begin{aligned}
\sum_{j=2}^{T} y_{j-1}\, y_j &= \sum_{j=1}^{N} \sum_{k=0}^{m-1} y_{(j-1)m+k}\, y_{(j-1)m+k+1} \\
&= \sum_{j=1}^{N} \sum_{k=1}^{m-1} y_{(j-1)m+k}\, y_{(j-1)m+k+1} + \sum_{j=1}^{N} y_{(j-1)m}\, y_{(j-1)m+1} \\
&= \sum_{j=1}^{N} \sum_{k=1}^{m-1} \underset{\sim}{y}'_j e_k e'_{k+1} \underset{\sim}{y}_j + \sum_{j=1}^{N} \underset{\sim}{y}'_{j-1} e_m e'_1 \underset{\sim}{y}_j \\
&= \sum_{j=1}^{N} \underset{\sim}{y}'_j \left(\sum_{k=1}^{m-1} e_k e'_{k+1} + e_m e'_1 \right) \underset{\sim}{y}_j - \sum_{j=1}^{N} \underset{\sim}{\varepsilon}'_j e_m e'_1 \underset{\sim}{y}_j \,,
\end{aligned}
$$

which yields (7.114).

7.7 Note first that

$$
\begin{aligned}
\sum_{j=1}^{N} \underset{\sim}{\varepsilon}'_j e_m e'_1 \underset{\sim}{y}_j &= \sum_{j=1}^{N} \underset{\sim}{\varepsilon}'_j e_m e'_1 (\underset{\sim}{\varepsilon}_1 + \cdots + \underset{\sim}{\varepsilon}_j) \\
&= \sum_{j=1}^{N} \varepsilon_{jm} (\varepsilon_1 + \varepsilon_{m+1} + \cdots + \varepsilon_{(j-1)m+1}) \\
&= \sum_{j=1}^{N} \xi_j (\eta_1 + \eta_2 + \cdots + \eta_j),
\end{aligned}
$$

where $\xi_j = \varepsilon_{jm}$ and $\eta_j = \varepsilon_{(j-1)m+1}$. Since $\{\xi_j\}$ and $\{\eta_j\}$ are i.i.d. $(0, \sigma^2)$ sequences and are independent of each other, it follows from the weak convergence result in Chapter 3 that $\sum_{j=1}^{N} \underset{\sim}{\varepsilon}'_j e_m e'_1 y_j \,/\, N$ converges to a nondegenerate distribution. Therefore (7.115) holds.

7.8 Let us put $y_0 = 0$ and suppose first that $0 \le l < m$. Then

$$\begin{aligned}\sum_{j=l+1}^{T} y_{j-l}\, y_j &= \sum_{j=1}^{N-1}\sum_{k=0}^{m-1} y_{(j-1)m+k}\, y_{(j-1)m+k+l} + \sum_{k=0}^{m-l} y_{(N-1)m+k}\, y_{(N-1)m+k+l} \\ &= \sum_{j=1}^{N-1} \underset{\sim}{y}'_j H_l\, \underset{\sim}{y}_j + R_N \\ &= \sum_{j,k=1}^{N-1} \min(N-j, N-k)\, \underset{\sim}{\varepsilon}'_j H_l\, \underset{\sim}{\varepsilon}_k + R_N,\end{aligned}$$

where R_N/T^2 converges in probability to 0. Thus (7.117) holds because of the same reasoning as in (7.116). When $l \ge m$, we may put $l = im + n$ ($i = 1, 2, \cdots;\ n = 0, 1, \cdots, m-1$). Since

$$y_j = y_{j-im} + \sum_{k=0}^{i-1} \varepsilon_{j-km}\,,$$

we have

$$\begin{aligned}\frac{1}{T^2}\sum_{j=l+1}^{T} y_{j-l}\, y_j &= \frac{1}{T^2}\sum_{j=l+1}^{T} y_{j-im-n}\, y_{j-im} + \frac{1}{T^2}\sum_{j=l+1}^{T}\sum_{k=0}^{i-1} y_{j-l}\, \varepsilon_{j-km} \\ &= \frac{1}{T^2}\sum_{j=n+1}^{T} y_{j-n}\, y_j + o_p(1)\end{aligned}$$

so that (7.117) holds for general l.

7.9 Defining $dX_M(u) = -\beta X_M(u)du + dw(u)$ with $X_M(0) = 0$, Girsanov's theorem gives

$$\begin{aligned}&E\left[\exp\left\{\frac{\theta_1}{M}\int_0^M Z_M(u)dZ_M(u) + \frac{\theta_2}{M^2}\int_0^M Z_M^2(u)du\right\}\right] \\ &= \exp\left\{M\left(-\frac{\beta-c}{2} - \frac{\theta_1}{2M}\right)\right\} E\left[\exp\left\{\left(\frac{\beta-c}{2} + \frac{\theta_1}{2M}\right) X_M^2(M)\right\}\right],\end{aligned}$$

where $\beta = \sqrt{c^2 - 2\theta_2/M^2}$ and $X_M(M) \sim \mathrm{N}(0, (1-e^{-2\beta M})/(2\beta))$. Then this m.g.f. is shown to be identical with that given in (7.121). Thus (7.123) is established.

7.10 It follows from Problem 7.9 and (7.121) that

$$\begin{aligned}&E\left[\exp\left\{\frac{\theta_1}{\sqrt{M}}\left(\int_0^M Z_M(u)dZ_M(u) + c\int_0^M Z_M^2(u)du\right) + \frac{\theta_2}{M}\int_0^M Z_M^2(u)du\right\}\right] \\ &= \exp\left\{\frac{cM - \sqrt{M}\theta_1}{2}\right\}\left[\cosh\nu + (cM - \sqrt{M}\theta_1)\frac{\sinh\nu}{\nu}\right]^{-\frac{1}{2}},\end{aligned}$$

where

$$\begin{aligned}\nu &= (c^2M^2 - 2cM\sqrt{M}\theta_1 - 2M\theta_2)^{\frac{1}{2}} \\ &= cM - \sqrt{M}\theta_1 - \frac{\theta_2}{c} - \frac{\theta_1^2}{2c} + O\left(\frac{1}{\sqrt{M}}\right).\end{aligned}$$

Then the above m.g.f. converges to $\exp\{(\theta_1^2/2 + \theta_2)/(2c)\}$, which implies (7.126). Similarly we can show that

$$\begin{aligned}&E\left[\exp\left\{\frac{\theta_1}{M}\int_0^M Z_M(u)dZ_M(u) + \frac{\theta_2}{M}\int_0^M Z_M^2(u)du\right\}\right] \\ &\qquad\longrightarrow \exp\left\{-\frac{1}{2}\left(\theta_1 - \frac{\theta_2}{c}\right)\right\},\end{aligned}$$

which implies (7.125).

7.11 It follows from (7.121) and (7.124) that

$$\begin{aligned}&E\left[\exp\left\{ce^{cM}\theta_1\left(\int_0^M Z_M(u)dZ_M(u) + c\int_0^M Z_M^2(u)du\right) + 2c^2e^{2cM}\theta_2\int_0^M Z_M^2(u)du\right\}\right] \\ &= \exp\left\{\frac{cM}{2}\left(1 - e^{cM}\theta_1\right)\right\}\left[\cosh\nu + cM\left(1 - e^{cM}\theta_1\right)\frac{\sinh\nu}{\nu}\right]^{-\frac{1}{2}},\end{aligned}$$

where

$$\begin{aligned}\frac{1}{\nu} &= \left(c^2M^2 - 2c^2M^2e^{cM}\theta_1 - 4c^2M^2e^{2cM}\theta_2\right)^{-\frac{1}{2}} \\ &= -\frac{1}{cM}\left(1 + e^{cM}\theta_1 + 2e^{2cM}\theta_2 + \frac{3}{2}e^{2cM}\theta_1^2 + O(e^{3cM})\right).\end{aligned}$$

Then we have

$$\cosh\nu + cM(1 - e^{cM}\theta_1)\frac{\sinh\nu}{\nu} = e^{cM}(1 - \theta_2 - \frac{1}{4}\theta_1^2) + O(e^{2cM}),$$

so that the above m.g.f. converges to $(1 - \theta_2 - \theta_1^2/4)^{-\frac{1}{2}}$, which is the joint m.g.f. of $(XY/2, X^2/2)$, where $(X, Y)' \sim \mathrm{N}(0, I_2)$. Thus (7.127) is established.

8.1 Putting $\varepsilon_j = 0$ for $j \leq 0$ we have

$$y_j = \frac{\varepsilon_j}{(1 - e^{i\theta}L)(1 - e^{-i\theta}L)} = \frac{1}{2i\sin\theta}\left[\frac{e^{i\theta}}{1 - e^{i\theta}L} - \frac{e^{-i\theta}}{1 - e^{-i\theta}L}\right]\varepsilon_j$$

$$
\begin{aligned}
&= \frac{e^{i\theta}}{2i\sin\theta}\left[e^{i(j-1)\theta}\varepsilon_1 + e^{i(j-2)\theta}\varepsilon_2 + \cdots + e^{i\theta}\varepsilon_{j-1} + \varepsilon_j\right] \\
&\quad -\frac{e^{-i\theta}}{2i\sin\theta}\left[e^{-i(j-1)\theta}\varepsilon_1 + e^{-i(j-2)\theta}\varepsilon_2 + \cdots + e^{-i\theta}\varepsilon_{j-1} + \varepsilon_j\right] \\
&= \frac{1}{\sin\theta}\sum_{k=1}^{j}\varepsilon_k \sin(j-k+1)\theta = \frac{1}{\sin\theta}\left[\, X_j \sin(j+1)\theta - Y_j \cos(j+1)\theta \,\right].
\end{aligned}
$$

8.2 It follows from (7.129) and (7.132) that

$$
\begin{aligned}
\sum_{j=2}^{T} y_{j-1}\, y_j &= \frac{1}{\sin^2\theta}\sum_{j=2}^{T}(X_{j-1}\sin j\theta - Y_{j-1}\cos j\theta) \\
&\qquad \times (X_j \sin(j+1)\theta - Y_j\cos(j+1)\theta) \\
&= \frac{1}{2\sin^2\theta}\sum_{j=2}^{T}\left[(X_{j-1}X_j + Y_{j-1}Y_j)\cos\theta + (X_{j-1}Y_j - X_jY_{j-1})\sin\theta\right. \\
&\qquad -(X_{j-1}X_j - Y_{j-1}Y_j)\cos(2j+1)\theta \\
&\qquad \left. -(X_{j-1}Y_j + X_jY_{j-1})\sin(2j+1)\theta\,\right] \\
&= \frac{\sigma^2 T\cos\theta}{4\sin^2\theta}\sum_{j=1}^{T} Z_T'\left(\frac{j}{T}\right) Z_T\left(\frac{j}{T}\right) + o_p(T^2),
\end{aligned}
$$

which implies (7.133), where we have used $X_j = X_{j-1} + \varepsilon_j \cos j\theta$ and $Y_j = Y_{j-1} + \varepsilon_j \sin j\theta$. Consider next

$$
\begin{aligned}
\sum_{j=2}^{T} y_{j-1}\,\varepsilon_j &= \frac{1}{\sin\theta}\sum_{j=2}^{T}(X_{j-1}\sin j\theta - Y_{j-1}\cos j\theta)\varepsilon_j \\
&= \frac{1}{\sin\theta}\sum_{j=2}^{T}(X_{j-1}(Y_j - Y_{j-1}) - Y_{j-1}(X_j - X_{j-1}))\,,
\end{aligned}
$$

which leads us to (7.134). Finally we have

$$
\begin{aligned}
\sum_{j=3}^{T} y_{j-2}\,\varepsilon_j &= \frac{1}{\sin\theta}\sum_{j=3}^{T}\left[\, X_{j-2}(\sin j\theta\cos\theta - \cos j\theta\sin\theta)\right. \\
&\qquad \left. -Y_{j-2}(\cos j\theta\cos\theta + \sin j\theta\sin\theta)\,\right]\varepsilon_j \\
&= \frac{1}{\sin\theta}\sum_{j=3}^{T}\left[\,\cos\theta\{X_{j-2}\Delta Y_j - Y_{j-2}\Delta X_j\}\right. \\
&\qquad \left. -\sin\theta\{X_{j-2}\Delta X_j + Y_{j-2}\Delta Y_j\}\,\right],
\end{aligned}
$$

which yields (7.135).

9.1 Putting $u_j = \varepsilon_j/\beta(L)$ it follows that, for any $k > 0$,

$$\begin{aligned} y_{j-k} &= \frac{u_{j-k}}{(1-L)^d} = \frac{u_{j-1} - (1-L)(u_{j-1} + \cdots + u_{j-k+1})}{(1-L)^d} \\ &= \frac{u_{j-1}}{(1-L)^d} - \frac{u_{j-1} + \cdots + u_{j-k+1}}{(1-L)^{d-1}}. \end{aligned}$$

Then it can be verified that

$$\frac{1}{T^{2d}\sigma^2} \sum_{j=p+1}^{T} \underset{\sim}{y}_{j-1} \underset{\sim}{y}'_{j-1} = \frac{1}{T^{2d}\sigma^2} \sum_{j=p+1}^{T} \left(\frac{u_{j-1}}{(1-L)^d} \right)^2 ee' + o_p(1)\,,$$

which establishes (7.140).

9.2 When $d = 1$, it holds that

$$\delta = M'\phi = \begin{pmatrix} 1 & 1 & & & & \\ & -1 & \cdot & & 0 & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & 0 & & & \cdot & 1 \\ & & & & & -1 \end{pmatrix} \phi\,,$$

from which we have $\delta_1 = \phi_1 + \phi_2$, $\delta_{q+1} = -\phi_{q+1}$ and $\delta_k = -\phi_k + \phi_{k+1}$ $(k = 2, \cdots, q)$. Solving for ϕ_k's we obtain (7.142). When $d = 2$, it holds that

$$\delta = M'\phi = \begin{pmatrix} 1 & 1 & 1 & & & & \\ & -1 & -2 & \cdot & & 0 & \\ & & 1 & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \cdot & \cdot & 1 \\ & 0 & & & & \cdot & -2 \\ & & & & & & 1 \end{pmatrix} \phi\,,$$

from which we have $\delta_1 = \phi_1 + \phi_2 + \phi_3$, $\delta_2 = -\phi_2 - 2\phi_3 + \phi_4$, $\delta_{q+1} = \phi_{q+1} - 2\phi_{q+2}$, $\delta_{q+2} = \phi_{q+2}$ and $\delta_k = \phi_k - 2\phi_{k+1} + \phi_{k+2}$ $(k = 3, \cdots, q)$, which yields (7.143).

9.3 Since $\{\Delta^k y_{i-1}\} \sim \mathrm{I}(d-k)$ and it holds that

$$\frac{1}{T^{d-k}\sqrt{T}} \sum_{j=p+1}^{T} \Delta^k y_{j-1} \Delta^d y_{j-m} \longrightarrow 0 \qquad \text{in probability}$$

for $k = 0, 1, \cdots, d-1$ and $m = 1, \cdots, q$, the off-block diagonal elements in the limiting distribution reduce to 0. Since $\{\underset{\sim}{u}_{j-1}\}$ is second-order stationary with i.i.d. innovations, we have, by the weak law of large numbers,

$$\frac{1}{T} \sum_{j=p+1}^{T} \underset{\sim}{u}_{j-1} \underset{\sim}{u}'_{j-1} \longrightarrow \Gamma \qquad \text{in probability.}$$

Finally the FCLT and the continuous mapping theorem gives the joint weak convergence :

$$\mathcal{L}\left(G_T^{-1} \sum_{j=p+1}^{T} \underset{\sim}{x}_{j-1} \underset{\sim}{x}'_{j-1} G_T^{-1} \right) \longrightarrow \mathcal{L}\left(\alpha^2 \sigma^2 F \right),$$

where $G_T = \text{diag}\,(T^d, \cdots, T) : \ d \times d$. Thus we obtain the conclusion.

9.4 In view of (7.145) we have only to show that $\bar{M}_1 Q_1 \underset{\sim}{y}_{j-1} = \underset{\sim}{z}_{j-1}$. It is easy to see that

$$\begin{aligned} \bar{M}_1 Q_1 \begin{pmatrix} y_{j-1} \\ \cdot \\ \cdot \\ \cdot \\ y_{j-p} \end{pmatrix} &= \bar{M}_1 \frac{1}{(1-L)^d} \begin{pmatrix} \varepsilon_{j-1} \\ \cdot \\ \cdot \\ \cdot \\ \varepsilon_{j-d} \end{pmatrix} \\ &= \left(\frac{\varepsilon_{j-1}}{(1-L)^d}, \cdots, \frac{\varepsilon_{j-1}}{1-L} \right)' = \underset{\sim}{z}_{j-1}\,. \end{aligned}$$

9.5 We first have

$$\begin{aligned} (\bar{M}'_1 G_T^{-1})^{-1} (\hat{\delta} - \delta) &= (G_T^{-1} \bar{M}_1 \sum_j \underset{\sim}{y}_{j-1} \underset{\sim}{y}'_{j-1} \bar{M}'_1 G_T^{-1})^{-1} G_T^{-1} \bar{M}_1 \sum_j \underset{\sim}{y}_{j-1} u_j \\ &= (G_T^{-1} \sum_j \underset{\sim}{x}_{j-1} \underset{\sim}{x}'_{j-1} G_T^{-1})^{-1} G_T^{-1} \sum_j \underset{\sim}{x}_{j-1} u_j\,, \end{aligned}$$

where

$$\begin{aligned} \underset{\sim}{x}_{j-1} &= (y_{j-1}, \Delta y_{j-1}, \cdots, \Delta^{d-1} y_{j-1})' \\ &= \left(\frac{u_{j-1}}{(1-L)^d}, \frac{u_{j-1}}{(1-L)^{d-1}}, \cdots, \frac{u_{j-1}}{1-L} \right)' \end{aligned}$$

with $u_j = \varepsilon_j \,/\, \beta(L)$. It is evident that

$$\mathcal{L}\left(\frac{1}{T}\sum_j \frac{u_{j-1}}{1-L}u_j\right) \longrightarrow \mathcal{L}\left(\alpha^2\sigma^2\left(\int_0^1 w(t)dw(t) + \frac{1-r}{2}\right)\right).$$

We show that, for $k = 2, \cdots, d$,

$$\text{(S9)} \qquad \mathcal{L}\left(\frac{1}{T^k}\sum_j \frac{u_{j-1}}{(1-L)^k}\,u_j\right) \longrightarrow \mathcal{L}\left(\alpha^2\sigma^2\int_0^1 F_{k-1}(t)dw(t)\right).$$

Let us consider the case of $k = 2$. Using the BN decomposition we have

$$\begin{aligned}
\sum_j \frac{u_{j-1}}{(1-L)^2}\,u_j &= \sum_j\left(\sum_{l=1}^{j-1}\sum_{m=1}^{l} u_m\right)u_j \\
&= \sum_j\sum_{l=1}^{j-1}\sum_{m=1}^{l}\left(\alpha\varepsilon_m + \tilde{\varepsilon}_{m-1} - \tilde{\varepsilon}_m\right)u_j \\
&= \alpha^2\sum_j \frac{\varepsilon_{j-1}}{(1-L)^2}\,\varepsilon_j + \alpha\sum_j \frac{\varepsilon_{j-1}}{(1-L)^2}(\tilde{\varepsilon}_{j-1} - \tilde{\varepsilon}_j) \\
&\quad + \tilde{\varepsilon}_0\sum_j (j-1)u_j - \sum_j \frac{\tilde{\varepsilon}_j}{1-L}\,u_j \\
&= \alpha^2\sum_j \frac{\varepsilon_{j-1}}{(1-L)^2}\,\varepsilon_j + O_p(T\sqrt{T}),
\end{aligned}$$

where we have used the fact that

$$\begin{aligned}
\sum_j \frac{\varepsilon_{j-1}}{(1-L)^2}(\tilde{\varepsilon}_{j-1} - \tilde{\varepsilon}_j) &= \sum_j \frac{\varepsilon_{j-1}}{(1-L)^2}\,\tilde{\varepsilon}_{j-1} - \sum_j\left(\frac{\varepsilon_j}{(1-L)^2} - \frac{\varepsilon_j}{1-L}\right)\tilde{\varepsilon}_j \\
&= \frac{\varepsilon_0}{(1-L)^2}\,\tilde{\varepsilon}_0 - \frac{\varepsilon_T}{(1-L)^2}\,\tilde{\varepsilon}_T + \sum_j \frac{\varepsilon_j}{1-L}\,\tilde{\varepsilon}_j \\
&= O_p(T\sqrt{T}).
\end{aligned}$$

Thus (S9) is established for $k = 2$. The case of $k \geq 3$ can also be proved by induction. Then we obtain (7.150) by the FCLT and the continuous mapping theorem.

9.6 Since

$$\begin{aligned}
T(\hat{\delta} - 1) &= \frac{1}{T^{2d-1}}\sum_{j=2}^{T} y_{j-1}(y_j - y_{j-1}) \Bigg/ \left[\frac{1}{T^{2d}}\sum_{j=2}^{T} y_{j-1}^2\right] \\
&= \frac{1}{2T^{2d-1}}\left[y_T^2 - \sum_{j=1}^{T}(y_j - y_{j-1})^2\right] \Bigg/ \left[\frac{1}{T^{2d}}\sum_{j=2}^{T} y_{j-1}^2\right]
\end{aligned}$$

and $\sum_{j=2}^{T}(y_j - y_{j-1})^2 = O_p(T^{2(d-1)})$, (7.151) follows from the FCLT and the continuous mapping theorem.

9.7 It follows from Theorem 4.2 that

$$
\begin{aligned}
E\left(e^{\theta Y_2}\right) &= E\left[\exp\left\{\theta x \int_0^1 F_1^2(t)dt - \frac{\theta}{2}F_1^2(1)\right\}\right] \\
&= E\left[\exp\left\{\theta x \int_0^1 X^2(t)dt - \frac{\theta}{2}X^2(1) - \beta\int_0^1 \frac{dX(t)}{dt}d\left(\frac{dX(t)}{dt}\right)\right.\right. \\
&\qquad \left.\left. + \frac{\beta^2}{2}\int_0^1 \left(\frac{dX(t)}{dt}\right)^2 dt\right\}\right],
\end{aligned}
$$

where $dX(t)/\,dt = \beta X(t) + w(t)$ and

$$
X(t) = e^{\beta t}\int_0^t e^{-\beta s}w(s)ds\,.
$$

Noting that

$$
\int_0^1 \frac{dX(t)}{dt}d\left(\frac{dX(t)}{dt}\right) = \frac{1}{2}\left\{(\beta X(1) + w(1))^2 - 1\right\},
$$

$$
\begin{aligned}
\int_0^1 \left(\frac{dX(t)}{dt}\right)^2 dt &= \int_0^1 (\beta X(t) + w(t))^2\,dt \\
&= \beta^2\int_0^1 X^2(t)dt + 2\beta\int_0^1 X(t)\left(dX(t) - \beta X(t)dt\right) + \int_0^1 w^2(t)dt \\
&= -\beta^2\int_0^1 X^2(t)dt + \beta X^2(1) + \int_0^1 w^2(t)dt\,,
\end{aligned}
$$

we obtain the first equality in (7.152), where $\beta = (2\theta x)^{\frac{1}{4}}$. Applying Girsanov's theorem again we arrive at the second equality, where $\gamma = i\beta$. The last equality is obvious.

Chapter 8.

1.1 Since we have $E(y_1^2) = \sigma^2$, $E(y_j^2) = (1+\alpha^2)\sigma^2$ and $E(y_j y_{j-1}) = -\alpha\sigma^2$ for $j \geq 2$, α can be uniquely determined as $\alpha = -E(y_j y_{j-1})/\,E(y_1^2)$, which may take any value. Note that, in the stationary case, we have $E(y_j^2) = (1+\alpha^2)\sigma^2$ for $j \geq 1$ and

$E(y_k y_{k-1}) = -\alpha\sigma^2$ for $k \geq 2$ so that the parameter vectors (α, σ^2) and $(1/\alpha, \alpha^2\sigma^2)$ give the same model.

1.2 It is easy to see that $y \sim \mathrm{N}(0, \sigma^2\Phi(\alpha))$, where $\Phi(\alpha) = \Omega(\alpha) - \alpha^2 e_1 e_1' = C^{-1}(\alpha)(C^{-1}(\alpha))'$ with $C(\alpha)$ defined in (8.36). Since $|\,C(\alpha)\,| = 1$ so that $\log|\,\Phi(\alpha)\,| = 0$, we arrive at (8.4).

1.3 Let us put $\Omega(\alpha) = (1+\alpha^2)I_T - 2\alpha B$. Then we have

$$D_T = |\,B - \lambda I_T\,| = -\lambda D_{T-1} - \frac{1}{4}D_{T-2}\,,$$

with $D_1 = -\lambda$ and $D_2 = \lambda^2 - 1/4$, from which we obtain

$$D_T = \frac{\sin(T+1)\theta}{2^T \sin\theta}\,, \ \cos\theta = -\lambda\,, \ \sin\theta = \sqrt{1-\lambda^2}\,, \qquad 0 < \theta < \pi\,.$$

Thus $D_T = 0$ yields $\theta = j\pi/(T+1)$ $(j = 1, \cdots, T)$ so that the eigenvalues of B are given by $\cos(j\pi/(T+1))$. Therefore the eigenvalues of $\Omega(\alpha)$ are given by $1 + \alpha^2 - 2\alpha\cos(j\pi/(T+1))$, $(j = 1, \cdots, T)$.

1.4 It is easy to obtain $D_T = |\,\Omega(\alpha)\,| = (1+\alpha^2)D_{T-1} - \alpha^2 D_{T-2}$ with $D_1 = 1 + \alpha^2$ and $D_2 = 1 + \alpha^2 + \alpha^4$. Then, if $|\,\alpha\,| \neq 1$, we derive $D_T = (1 - \alpha^{2(T+1)})\big/(1-\alpha^2)$. When $|\,\alpha\,| = 1$, we have $D_T = T + 1$.

2.1 Putting $\delta_{jT} = \cos(j\pi/(T+1))$ and noting that $y \sim \mathrm{N}(0, \sigma^2\Omega(\alpha_0))$, we deduce

$$\begin{aligned}
\mathcal{L}\left(\frac{1}{T}y'\Omega^{-1}(1)y\right) &= \mathcal{L}\left(\frac{\sigma^2}{T}Z'\Omega^{\frac{1}{2}}(\alpha_0)\Omega^{-1}(1)\Omega^{\frac{1}{2}}(\alpha_0)Z\right) \\
&= \mathcal{L}\left(\frac{\sigma^2}{T}\sum_{j=1}^{T}\frac{(1-\alpha_0)^2 + 2\alpha_0(1-\delta_{jT})}{2(1-\delta_{jT})}Z_j^2\right) \\
&= \mathcal{L}\left(\frac{\alpha_0\sigma^2}{T}\sum_{j=1}^{T}Z_j^2 + o_p(1)\right),
\end{aligned}$$

which gives (8.12), where $\{Z_j\} \sim \mathrm{NID}(0,1)$. We also deduce

$$\frac{1}{T+1}\sum_{j=0}^{T}\alpha^{2j} \quad = \quad \frac{1}{T+1}\frac{1-\alpha^{2(T+1)}}{1-\alpha^2} = \frac{1 - \left(1 - \dfrac{\theta}{T}\right)^{2(T+1)}}{2\theta + O\left(\dfrac{1}{T}\right)}$$

$$\longrightarrow \quad \frac{1-e^{-2\theta}}{2\theta} = \frac{\sinh\theta}{\theta e^{\theta}}\,.$$

2.2 Let us consider

$$Y_T = \sum_{j=1}^{K_T} A_{jT} Z_j^2 + \sum_{j=K_T+1}^{T} A_{jT} Z_j^2\,,$$

where K_T is a sequence of integers such that $K_T \to \infty\,, K_T/T \to 0$ and $K_T^2/T \to \infty$ as $T \to \infty$, while

$$A_{jT} = \frac{c^2 + 4\left(T^2 - cT\right) s_{jT}^2}{4T^2 s_{jT}^2 \left(\theta^2 + 4\left(T^2 - \theta T\right) s_{jT}^2\right)}\,.$$

For $j = 1, \cdots, K_T$ it holds that

$$4\,(T+1)^2\, s_{jT}^2 = j^2\pi^2 + j^4 O\left(T^{-2}\right) = j^2\pi^2 \left(1 + O\left(\left(\frac{K_T}{T}\right)^2\right)\right)\,.$$

We also have

$$\begin{aligned}
\mathrm{P}\left(\sum_{j=K_T+1}^{T} A_{jT} Z_j^2 > \varepsilon\right) &< \frac{1}{\varepsilon} \sum_{j=K_T+1}^{T} A_{jT} \\
&< \frac{1}{\varepsilon}\left(T - K_T\right) A_{K_T T} \\
&= \frac{1}{\varepsilon}\, O\left(\frac{T}{K_T^2}\right) \to 0\,.
\end{aligned}$$

Then we can deduce that

$$\mathcal{L}\left(Y_T\right) \longrightarrow \mathcal{L}\left(\sum_{n=1}^{\infty} \frac{n^2\pi^2 + c^2}{n^2\pi^2\left(n^2\pi^2 + \theta^2\right)} Z_n^2\right)\,.$$

Since the second and third terms of S_{T1} in (8.14) converge in probability to θ and 0, respectively, we establish (8.15).

2.3 We have only to show that

$$\text{(S10)} \qquad \mathcal{L}\left(\frac{1}{\sigma^2} y'\left(\Omega^{-1}(\alpha) - \Omega^{-1}(1)\right) y\right) \longrightarrow \mathcal{L}\left(\int_0^1\!\!\int_0^1 \bar{K}_1(s,t;0)\,dw(s)\,dw(t) + \theta\right)\,,$$

where $\bar{K}_1 = -2K_1$. Noting that $y = -\alpha_0\varepsilon_0 e_1 + C^{-1}(\alpha_0)\varepsilon = D(\alpha_0)\varepsilon^*$, where $e_1 = (1, 0, \cdots, 0)' : T \times 1$, $D(\alpha_0) = (-\alpha_0 e_1, C^{-1}(\alpha_0))$, $\varepsilon^* = (\varepsilon_0, \varepsilon')'$ and $C(\alpha)$ is defined in (8.36), consider

$$\begin{aligned}\frac{1}{\sigma^2}y'\left(\Omega^{-1}(\alpha) - \Omega^{-1}(1)\right)y &= \frac{1}{\sigma^2}\varepsilon^{*\prime}D'(\alpha_0)\left(\Omega^{-1}(\alpha) - \Omega^{-1}(1)\right)D(\alpha_0)\varepsilon^* \\ &= \frac{1}{T}Z'B_T Z + \frac{\theta}{T}Z'Z + R_T\,,\end{aligned}$$

where $Z = \varepsilon/\sigma \sim \mathrm{N}(0, I_T)$ and

$$\begin{aligned}B_T &= T\left[\left(C^{-1}(\alpha_0)\right)'\left(\Omega^{-1}(\alpha) - \Omega^{-1}(1)\right)C^{-1}(\alpha_0)\right] - \theta I_T\,, \\ R_T &= \frac{1}{T\sigma^2}\alpha_0^2\varepsilon_0^2 e_1'\left(\Omega^{-1}(\alpha) - \Omega^{-1}(1)\right)e_1 \\ &\quad -\frac{2}{T\sigma^2}\alpha_0\varepsilon_0 e_1'\left(\Omega^{-1}(\alpha) - \Omega^{-1}(1)\right)C^{-1}(\alpha_0)\varepsilon\,.\end{aligned}$$

Using the fact that

$$\Omega^{-1}(\alpha) = C'(\alpha)C(\alpha) - \frac{C'(\alpha)d_\alpha d_\alpha' C(\alpha)}{1 + d_\alpha' d_\alpha}\,,$$

where $d_\alpha = (\alpha, \alpha^2, \cdots, \alpha^T)'$, it can be checked that $R_T \to 0$ in probability. Then Theorem 5.12 establishes (S10) after some algebra.

2.4 Let us consider

$$\begin{aligned}&E\left[\exp\left\{iu\left(X_1(\theta) + \frac{1}{2}\log\frac{\sinh\theta}{\theta}\right)\right\}\right] \\ &= \prod_{n=1}^{\infty}\left[1 - \frac{iu\theta^2(n^2\pi^2 + c^2)}{n^2\pi^2(n^2\pi^2 + \theta^2)}\right]^{-\frac{1}{2}} = \prod_{n=1}^{\infty}\left[\frac{1 + \dfrac{(1 - iu)\theta^2}{n^2\pi^2} - \dfrac{ic^2\theta^2 u}{n^4\pi^4}}{1 + \dfrac{\theta^2}{n^2\pi^2}}\right]^{-\frac{1}{2}} \\ &= \prod_{n=1}^{\infty}\left[\frac{\left(1 - \dfrac{a(u) + b(u)}{n^2\pi^2}\right)\left(1 - \dfrac{a(u) - b(u)}{n^2\pi^2}\right)}{1 + \dfrac{\theta^2}{n^2\pi^2}}\right]^{-\frac{1}{2}},\end{aligned}$$

which leads us to (8.18).

2.5 We first have

$$\frac{dg_{T1}(\alpha)}{d\alpha} = \frac{T}{2}\,\frac{y'\Omega^{-1}(\alpha)\dfrac{d\Omega(\alpha)}{d\alpha}\Omega^{-1}(\alpha)y}{y'\Omega^{-1}(\alpha)y} - \frac{1}{2}\mathrm{tr}\left(\Omega^{-1}(\alpha)\frac{d\Omega(\alpha)}{d\alpha}\right).$$

Noting that $d\Omega(\alpha)/d\alpha\,|_{\alpha=1} = \Omega(1)$ and $d\Omega(\alpha)/d\alpha\,|_{\alpha=-1} = -\Omega(-1)$ we obtain the conclusion.

2.6 When $l=1$, we have

$$\frac{dh_{T1}(\theta)}{d\theta} = \frac{1}{2}\,\frac{y'\dfrac{d\Omega^{-1}(\alpha)}{d\alpha}y}{y'\Omega^{-1}(\alpha)y} + \frac{1}{T}\,\frac{\sum_{i=1}^{T} i\alpha^{2i-1}}{\sum_{i=0}^{T}\alpha^{2i}}.$$

We first have

$$\begin{aligned}
\frac{1}{T^2}\sum_{i=1}^{T} i\alpha^{2i-1} &= \frac{1}{2T^2}\,\frac{d}{d\alpha}\sum_{i=1}^{T}\alpha^{2i} \\
&= \frac{1}{2T^2}\,\frac{-2(T+1)(1-\alpha^2)\alpha^{2T+1} + 2\alpha(1-\alpha^{2(T+1)})}{(1-\alpha^2)^2} \\
&\longrightarrow \frac{1-(2\theta+1)e^{-2\theta}}{4\theta^2},
\end{aligned}$$

which yields, using (8.13),

$$\begin{aligned}
\frac{1}{T}\,\frac{\sum_{i=1}^{T} i\alpha^{2i-1}}{\sum_{i=0}^{T}\alpha^{2i}} &\longrightarrow \frac{1-(2\theta+1)e^{-2\theta}}{4\theta^2} \times \frac{\theta e^{\theta}}{\sinh\theta} \\
&= \frac{1}{2}\left(1+\frac{1}{\theta}-\coth\theta\right).
\end{aligned}$$

Putting $\delta_j = \cos(j\pi/\,(T+1))$ and $s_j = \sin(j\pi/\,2(T+1))$ we consider

$$\begin{aligned}
\frac{1}{T\sigma^2}\,y'\,\frac{d\Omega^{-1}(\alpha)}{d\alpha}\,y &= -\frac{1}{T}\sum_{j=1}^{T}\frac{(1+\alpha_0^2-2\alpha_0\delta_j)(2\alpha-2\delta_j)}{(1+\alpha^2-2\alpha\delta_j)^2}\,Z_j^2 \\
&= -\frac{1}{T}\sum_{j=1}^{T}\frac{\left(c^2+4(T^2-cT)s_j^2\right)(4T^2s_j^2-2\theta T)}{\left(\theta^2+4(T^2-\theta T)s_j^2\right)^2}\,Z_j^2
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{T} \frac{2\theta\left(c^2+4(T^2-cT)s_j^2\right)}{\left(\theta^2+4(T^2-\theta T)s_j^2\right)^2} Z_j^2 - \frac{1}{T}\sum_{j=1}^{T} Z_j^2 \\
&\quad -\frac{1}{T}\sum_{j=1}^{T}\left[\frac{4T^2 s_j^2\left(c^2+4(T^2-cT)s_j^2\right)}{\left(\theta^2+4(T^2-\theta T)s_j^2\right)^2} - 1\right] Z_j^2 \,,
\end{aligned}
$$

which is shown to converge in distribution to

$$
\sum_{n=1}^{\infty} \frac{2\theta(n^2\pi^2+c^2)}{(n^2\pi^2+\theta^2)^2} Z_n^2 - 1 \,.
$$

Since $y'\Omega^{-1}(\alpha)y\big/T \to \sigma^2$ in probability, we have proved (8.21) for $l=1$. The case of $l=2$ can be proved similarly.

2.7 Since the c.f. of $\sum_{n=1}^{\infty} Z_n^2\big/(n^2\pi^2)$ is given by $\left(\left(\sin\sqrt{2i\theta}\right)\big/\sqrt{2i\theta}\right)^{-1/2}$, we immediately obtain (8.26) due to Theorem 5.13.

2.8 The first and second derivatives of

$$
\log\sinh\theta = \log\theta + \sum_{n=1}^{\infty}\log\left(\frac{n^2\pi^2+\theta^2}{n^2\pi^2}\right)
$$

with respect to θ are :

$$
\begin{aligned}
\coth\theta &= \frac{1}{\theta} + \sum_{n=1}^{\infty}\frac{2\theta}{n^2\pi^2+\theta^2}\,, \\
-\mathrm{cosech}^2\,\theta &= -\frac{1}{\theta^2} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi^2+\theta^2} - \sum_{n=1}^{\infty}\frac{4\theta^2}{(n^2\pi^2+\theta^2)^2}
\end{aligned}
$$

Thus we obtain

$$
\sum_{n=1}^{\infty}\frac{\theta^3}{(n^2\pi^2+\theta^2)^2} = \frac{\theta}{4}\mathrm{cosech}^2\theta + \frac{1}{4}\coth\theta - \frac{1}{2\theta}\,,
$$

which yields (8.30).

2.9 It is easy to deduce

$$
\phi(\theta;x) = \prod_{n=1}^{\infty}\left[1-\frac{2i\theta(n^2\pi^2+c^2)}{(n^2\pi^2+x^2)^2}\right]^{-\frac{1}{2}}
$$

$$
\begin{aligned}
&= \prod_{n=1}^{\infty}\left[\frac{\left(1-\dfrac{c(\theta)+d(\theta)}{n^2\pi^2}\right)\left(1-\dfrac{c(\theta)-d(\theta)}{n^2\pi^2}\right)}{\left(1+\dfrac{x^2}{n^2\pi^2}\right)^2}\right]^{-\frac{1}{2}} \\
&= \frac{\sinh x}{x}\left[\frac{\sin\sqrt{c(\theta)+d(\theta)}}{\sqrt{c(\theta)+d(\theta)}}\,\frac{\sin\sqrt{c(\theta)-d(\theta)}}{\sqrt{c(\theta)-d(\theta)}}\right]^{-\frac{1}{2}} .
\end{aligned}
$$

3.1 Let G_{jk} be the (j,k)-th element of $(C(1)C^{-1}(\alpha_0))'C(1)C^{-1}(\alpha_0)$. Then we have

(S11) $\qquad G_{jj} = 1+(1-\alpha_0)^2(T-j), \qquad G_{jk} = 1-\alpha_0+(1-\alpha_0)^2(T-k)\,, \quad (j<k)\,.$

Thus it holds that

$$
\frac{1}{T}\sum_{j,k=1}^{T} G_{jk}\varepsilon_j\varepsilon_k = \frac{1}{T}\left[\sum_{j=1}^{T}\left\{1+\frac{c^2}{T^2}(T-j)\right\}\varepsilon_j^2 + 2\sum_{j<k}\left\{\frac{c}{T}+\frac{c^2}{T^2}(T-k)\right\}\varepsilon_j\varepsilon_k\right],
$$

which clearly converges in probability to σ^2.

3.2 Let H_{jk} be the (j,k)-th element of $\left(C(\alpha)C^{-1}(\alpha_0)\right)'C(\alpha)C^{-1}(\alpha_0)$. Then we have

(S12) $\qquad H_{jj} = 1+(\alpha-\alpha_0)^2\,\dfrac{1-\alpha^{2(T-j)}}{1-\alpha^2}\,,$

(S13) $\qquad H_{jk} = (\alpha-\alpha_0)\alpha^{k-j-1}+(\alpha-\alpha_0)^2\,\dfrac{\alpha^{k-j}-\alpha^{2T-j-k}}{1-\alpha^2}\,, \quad (j<k)\,.$

Thus, using (S11), we obtain

$$
\begin{aligned}
B_T(j,j) &= T(H_{jj}-G_{jj}) = -\frac{c^2}{T}(T-j)+\frac{(c-\theta)^2}{T}\,\frac{1-\alpha^{2(T-j)}}{1-\alpha^2}\,, \\
B_T(j,k) &= T(H_{jk}-G_{jk}) \\
&= -c-\frac{c^2}{T}(T-k)+(c-\theta)\alpha^{k-j-1} \\
&\quad +\frac{(c-\theta)^2}{T}\,\frac{\alpha^{k-j}-\alpha^{2T-j-k}}{1-\alpha^2}\,, \quad (j<k).
\end{aligned}
$$

If we replace $B_T(j,j)$ by $B_T(j,j)-\theta$, then we can find the uniform limit of the modified $B_T(j,k)$ as $-2K_2(s,t;\theta)$ with K_2 defined in (8.40). It follows from Theorem 5.12 that

$$
\mathcal{L}(S_{T2}) \longrightarrow \mathcal{L}\left(-2\int_0^1\int_0^1 K_2(s,t;\theta)dw(s)dw(t)+\theta\right),
$$

which yields (8.38).

3.3 We consider the integral equation :

$$\begin{aligned} f(t) &= \lambda \int_0^1 K_2(s,t;\theta) f(s) ds \\ &= \frac{\lambda\theta}{2}\left[\cosh\theta(1-t)\int_0^t e^{-\theta(1-s)} f(s) ds + e^{-\theta(1-t)}\int_t^1 \cosh\theta(1-s) f(s) ds\right], \end{aligned}$$

which is shown to be equivalent to

$$f''(t) - \theta^2\left(1-\frac{\lambda}{2}\right) f(t) = 0\,, \qquad f'(0) = \theta f(0)\,, \qquad f'(1) = 0\,.$$

Suppose first that $\lambda \neq 2$. Then the general solution is

$$f(t) = c_1 e^{At} + c_2 e^{-At}\,, \qquad A = \theta\sqrt{1-\frac{\lambda}{2}}\,.$$

The two boundary conditions yield $M(\lambda)c = 0$, where $c = (c_1, c_2)'$ and

$$M(\lambda) = \begin{pmatrix} A-\theta & -A-\theta \\ Ae^{A} & -Ae^{-A} \end{pmatrix}.$$

Since $|\, M(\lambda)\,| = 2A(\theta\cosh A + A\sinh A)$, the candidate for the FD is given as in (8.42). When $\lambda = 2$, we have $f''(t) = 0$ with $f'(0) = \theta f(0)$ and $f'(1) = 0$. This implies that $\theta = 0$, which is a contradiction. Checking the second condition in Theorem 5.5, we can conclude that (8.42) is the FD of $K_2(s,t;\theta)$ in (8.41).

3.4 Let us consider

$$\frac{dh_{T2}(\theta)}{d\theta} = \frac{1}{2}\frac{d}{d\alpha}\log y'\Phi^{-1}(\alpha)y = \frac{1}{2}\frac{\varepsilon'\dfrac{dH(\alpha)}{d\alpha}\varepsilon}{y'\Phi^{-1}(\alpha)y}\,,$$

where $H(\alpha) = (C(\alpha)C^{-1}(\alpha_0))'\,C(\alpha)C^{-1}(\alpha_0)$. It can be shown as in (8.37) that $y'\,\Phi^{-1}(\alpha)\,y\big/T \to \sigma^2$ in probability. Let the (j,k)-th element of $dH(\alpha)/\,d\alpha$ be F_{jk}. Using (S12) and (S13) we obtain

$$F_{jj} = \left(\frac{2(\alpha-\alpha_0)}{1-\alpha^2} + \frac{2\alpha(\alpha-\alpha_0)^2}{(1-\alpha^2)^2}\right)\left(1-\alpha^{2(T-j)}\right)$$

$$
\begin{aligned}
& +\frac{(\alpha-\alpha_0)^2}{1-\alpha^2}\left(-2(T-j)\right)\alpha^{2T-2j-1}\,, \\
F_{jk} &= \left(\alpha+(\alpha-\alpha_0)(k-j-1)\right)\alpha^{k-j-2} \\
& +\left(\frac{2(\alpha-\alpha_0)}{1-\alpha^2}+\frac{2\alpha(\alpha-\alpha_0)^2}{(1-\alpha^2)^2}\right)\left(\alpha^{k-j}-\alpha^{2T-j-k}\right) \\
& +\frac{(\alpha-\alpha_0)^2}{1-\alpha^2}\left((k-j)\alpha^{k-j-1}-(2T-j-k)\alpha^{2T-j-k-1}\right)\,, \quad (j<k)\,.
\end{aligned}
$$

Then it holds that

$$
\begin{aligned}
\mathcal{L}\left(\frac{1}{T\sigma^2}\varepsilon'\frac{dH(\alpha)}{d\alpha}\varepsilon\right) &= \mathcal{L}\left(\frac{1}{T\sigma^2}\varepsilon'\left(\frac{dH(\alpha)}{d\alpha}+I_T\right)\varepsilon-\frac{1}{T\sigma^2}\varepsilon'\varepsilon\right) \\
&\longrightarrow \mathcal{L}\left(\int_0^1\int_0^1 J(s,t;\theta)dw(s)dw(t)-1\right)\,,
\end{aligned}
$$

where $J(s,t;\theta)=2\partial K_2(s,t;\theta)/\,\partial\theta$. Thus (8.44) is established for $l=1$.

3.5 Consider the integral equation :

$$
f(t)=\lambda\int_0^1\left\{\frac{1}{2}+c-\frac{c}{2}(s+t)+\frac{c^2}{2}(1-s)(1-t)\right\}f(s)ds\,,
$$

which is equivalent to $f(t)=a+bt$ with

$$
\begin{aligned}
f(0) &= \frac{\lambda}{2}\int_0^1(1+2c+c^2-cs-c^2s)f(s)ds=a\,, \\
f(1) &= \frac{\lambda}{2}\int_0^1(1+c-cs)f(s)ds=a+b\,.
\end{aligned}
$$

Then $\lambda(\neq 0)$ is an eigenvalue if and only if

$$
\begin{aligned}
|\,M(\lambda)\,| &= \left|\begin{pmatrix} \lambda\left(\frac{1}{2}+\frac{3c}{4}+\frac{c^2}{4}\right)-1 & \lambda\left(\frac{1}{4}+\frac{c}{3}+\frac{c^2}{12}\right) \\ \lambda\left(\frac{1}{2}+\frac{c}{4}\right)-1 & \lambda\left(\frac{1}{4}+\frac{c}{12}\right)-1 \end{pmatrix}\right| \\
&= 1-\frac{\lambda}{6}(c^2+3c+3)\,,
\end{aligned}
$$

which is the FD of $dX_2(\theta)/d\theta\,|_{\theta=0}+\dfrac{1}{2}$. Thus (8.48) is established.

3.6 Let us consider the integral equation :

$$
f(t)=\lambda\int_0^1\left.\frac{\partial K_2(s,t;\theta)}{\partial\theta}\right|_{\theta=-x}f(s)ds\,,
$$

which is equivalent to $f''''(t) - (2x^2 + \lambda x)f''(t) + x^4 f(t) = 0$ with $f'(1) = f'''(1) = 0$ and

$$\begin{aligned} f(1) &= \frac{\lambda}{2}\int_0^1 (1 + x - xs)e^{x(1-s)} f(s)ds\,, \\ f'''(0) &= -xf''(0) + (x^2 + \lambda x)f'(0) + (x^3 + \lambda x^2)f(0)\,. \end{aligned}$$

The general solution is given by

$$f(t) = c_1 e^{At} + c_2 e^{-At} + c_3 e^{Bt} + c_4 e^{-Bt}\,,$$

where

$$A^2 = \frac{x}{2}\left(\lambda + 2x + \sqrt{\lambda^2 + 4\lambda x}\right)\,, \qquad B^2 = \frac{x}{2}\left(\lambda + 2x - \sqrt{\lambda^2 + 4\lambda x}\right)\,.$$

We then have $M(\lambda)c = 0$, where $c = (c_1, c_2, c_3, c_4)'$ and $M(\lambda)$ is a 4×4 matrix constructed from the four boundary conditions given above. Evaluating $|\,M(\lambda)\,|$ by REDUCE we obtain (8.50).

4.1 Noting that $y \sim \mathrm{N}\,(0, \sigma^2(\Omega(\alpha_{m0}) \otimes I_m))$ and $|\,\Omega(\alpha_m) \otimes I_m\,| = |\,\Omega(\alpha_m)\,|^m$ we obtain easily the log-likelihood for α_m and σ^2, from which (8.57) results by concentrating σ^2 out.

4.2 Noting that

$$E\left[\exp\left\{i\theta\sum_{n=1}^{\infty}\frac{n^2\pi^2 + c^2}{(n^2\pi^2 + x^2)^2}\,\underset{\sim}{Z}'_n\,\underset{\sim}{Z}_n\right\}\right] = \prod_{n=1}^{\infty}\left[1 - \frac{2i\theta(n^2\pi^2 + c^2)}{(n^2\pi^2 + x^2)^2}\right]^{-\frac{m}{2}}\,,$$

(8.63) can be easily established because of (8.33).

4.3 Let us consider

$$\frac{1}{N}Z'\left\{(B - \theta I_N) \otimes I_m\right\}Z = \frac{1}{N}\sum_{j,k=1}^{N} A_{jk}Z_j'Z_k\,,$$

where $\{Z_j\} \sim \mathrm{NID}(0, I_m)$. The weak convergence of this quantity for $m = 1$ has already been established in (8.38). Noting that $Z'Z/N \to m$ in probability, (8.65) follows from (5.154).

4.4 It follows from (5.155) that

$$E\left[\exp\left\{i\theta\int_0^1\int_0^1 \frac{\partial K_2(s,t;\theta)}{\partial\theta}\bigg|_{\theta=0} d\underset{\sim}{w}'(s)d\underset{\sim}{w}(t)\right\}\right] = (D(2i\theta))^{-\frac{m}{2}},$$

where $D(\lambda)$ is the FD of $\partial K_2(s,t;\theta)/\partial\theta\,|_{\theta=0}$. We have already obtained $D(\lambda) = 1-\lambda(c^2+3c+3)/6$, which establishes (8.68).

4.5 Suppose that $c=0$. Then it holds that

$$\frac{dX_{m2}(\theta)}{d\theta} = \int_0^1\int_0^1 \frac{\partial K_2(s,t;\theta)}{\partial\theta} d\underset{\sim}{w}'(s)d\underset{\sim}{w}(t),$$

where $\partial K_2(s,t;\theta)/\partial\theta$ is defined by (8.46) with $c=0$. Then we obtain

$$\begin{aligned} E\left(\frac{dX_{m2}(\theta)}{d\theta}\right) &= -\frac{m}{4}(1-e^{-2\theta}) < 0, \\ V\left(\frac{dX_{m2}(\theta)}{d\theta}\right) &= \frac{m}{16\theta} + O\left(\frac{1}{\theta^2}\right). \end{aligned}$$

Thus the consistency proof for $m=1$ can be used in the present case.

4.6 Note first that

$$E\left[\exp\left\{i\theta\int_0^1\int_0^1 \frac{\partial K_2(s,t;\theta)}{\partial\theta}\bigg|_{\theta=-x} d\underset{\sim}{w}'(s)d\underset{\sim}{w}(t)\right\}\right] = (D(2i\theta))^{-\frac{m}{2}},$$

where $D(\lambda)$ is the FD of $\partial K_2(s,t;\theta)/\partial\theta\,|_{\theta=-x}$. Then (8.71) can be established because of (8.50).

4.7 It is easy to obtain

$$\frac{d^2X_{M1}(\theta)}{d\theta^2}\bigg|_{\theta=0} = M^2\left[\sum_{n=1}^{\infty}\left(\frac{1}{n^2\pi^2}+\frac{c^2M^2}{n^4\pi^4}\right)Z_n^2 - \frac{1}{6}\right].$$

Thus (8.77) follows from (8.25) and (8.26). We can prove (8.78) similarly.

5.1 Let us put

$$\frac{1}{T}\,y'\Omega^{-1}(1)y = \frac{1}{T}\,u'A_Tu + R_T,$$

where $A_T = \left(C^{-1}(\alpha_0)\right)' \left[C'(1)C(1) - C'(1)ee'C(1)/(T+1)\right] C^{-1}(\alpha_0)$ and

$$\begin{aligned} R_T &= \frac{1}{T}\alpha_0^2 u_0^2 e_1' \left[C'(1)C(1) - \frac{1}{T+1}C'(1)ee'C(1)\right] e_1 \\ &\quad -\frac{2}{T}\alpha_0 u_0 e_1' \left[C'(1)C(1) - \frac{1}{T+1}C'(1)ee'C(1)\right] C^{-1}(\alpha_0)u \\ &= \frac{\alpha_0^2 u_0^2}{T+1} - \frac{2\alpha_0 u_0}{T(T+1)} \sum_{j=1}^{T} \left((T-j)(1-\alpha_0)+1\right) u_j \\ &= o_p(1)\,. \end{aligned}$$

By the weak law of large numbers it holds that

$$\begin{aligned} &\frac{1}{T(T+1)} u' \left(C^{-1}(\alpha_0)\right)' C'(1)ee'C(1)C^{-1}(\alpha_0)u \\ &\quad = \frac{1}{T(T+1)} \left\{ \sum_{j=1}^{T} \left(c\left(1-\frac{j}{T}\right)+1\right) u_j \right\}^2 \\ &\quad \longrightarrow 0 \qquad \text{in probability}\,. \end{aligned}$$

Thus, using (S11), we have

$$\begin{aligned} \frac{1}{T}\, y'\Omega^{-1}(1)y &= \frac{1}{T}\sum_{j,k=1}^{T} G_{jk} u_j u_k + o_p(1) \\ &= \frac{1}{T}\left[\sum_{j=1}^{T} \left(1+\frac{c^2}{T^2}(T-j)\right) u_j^2 \right. \\ &\qquad\qquad \left. +2\sum_{j<k} \left(\frac{c}{T}+\frac{c^2}{T^2}(T-k)\right) u_j u_k \right] + o_p(1) \\ &= \frac{1}{T}\sum_{j=1}^{T} u_j^2 + o_p(1)\,, \end{aligned}$$

which yields (8.81).

5.2 The first equality can be proved by noting that $y = -\alpha_0 u_0 e_1 + C^{-1}(\alpha_0)u$. Since $u'u/T \to \sigma^2 \sum_{l=0}^{\infty} \phi_l^2$ in probability, (8.82) follows from (8.17) and (5.161).

5.3 It is easy to establish (8.87) by using (S11) and the weak law of large numbers. We can also prove (8.88) using (8.38) and (5.161).

6.1 It follows from (8.57) that

$$\left.\frac{d^2 l_{T1}(\alpha)}{d\alpha^2}\right|_{\alpha=1} = T\left(\frac{y'\left(\Omega^{-2}(1)\otimes I_m\right)y}{y'\left(\Omega^{-1}(1)\otimes I_m\right)y} - \frac{1}{2}\right) - \frac{m(N^2-N)}{6}.$$

Then it holds that

$$P\left(\left.\frac{d^2 l_{T1}(\alpha)}{d\alpha^2}\right|_{\alpha=1} < 0\right) = P\left(\frac{N+2}{6}y'\left(\Omega^{-1}(1)\otimes I_m\right)y - y'\left(\Omega^{-2}(1)\otimes I_m\right)y > 0\right),$$

which leads us to (8.91).

6.2 Noting that $\Phi^{-1}(1) = C'C$ and $d\Phi(\alpha)/\,d\alpha\,|_{\alpha=1} = (C'C)^{-1} - e_1 e_1'$ with $C = C(1)$ and $e_1 = (1, 0, \cdots, 0)' : N \times 1$, we obtain, from (8.64),

$$P\left(\left.\frac{dl_{T2}(\alpha)}{d\alpha}\right|_{\alpha=1} > 0\right) = P\left(\varepsilon'(A_N \otimes I_m)\varepsilon > 0\right),$$

where

$$\begin{aligned} A_N &= \left(C^{-1}(\alpha_0)\right)'\left[C'C\left((C'C)^{-1} - e_1e_1'\right)C'C\right]C^{-1}(\alpha_0) \\ &= \left(CC^{-1}(\alpha_0)\right)' CC^{-1}(\alpha_0) - \left(CC^{-1}(\alpha_0)\right)' ee'CC^{-1}(\alpha_0), \end{aligned}$$

with $e = (1, \cdots, 1)' : N \times 1$. Then (8.92) follows from (S11) and the fact that the j-th component of $\left(CC^{-1}(\alpha_0)\right)' e$ is given by $(N-j)(1-\alpha_0)+1$.

7.1 The first two equalities are obvious. Let us consider

$$\begin{aligned} &\sum_{j=1}^{T}(1+\rho_0\lambda_{jT})\left(\frac{1}{1+\rho\lambda_{jT}} - 1\right)Z_j^2 \\ &\quad = -\theta^2\sum_{j=1}^{T}\frac{c^2+4T^2s_{jT}^2}{4T^2s_{jT}^2\left(\theta^2+4T^2s_{jT}^2\right)}Z_j^2, \end{aligned}$$

where $s_{jT} = \sin\left(\left(j-\frac{1}{2}\right)\pi\Big/(2T+1)\right)$. Using the same arguments as in the solution to Problem 2.2 we can establish (8.100).

7.2 We first obtain

$$\frac{dY_1(\theta)}{d\theta} = \sum_{n=1}^{\infty}\frac{\theta\left(\left(n-\frac{1}{2}\right)^2\pi^2 + c^2\right)}{\left(\left(n-\frac{1}{2}\right)^2\pi^2+\theta^2\right)^2}Z_n^2 - \sum_{n=1}^{\infty}\frac{\theta}{\left(n-\frac{1}{2}\right)^2\pi^2+\theta^2},$$

which yields

$$E\left(\frac{dY_1(\theta)}{d\theta}\right) = -\sum_{n=1}^{\infty}\frac{\theta(\theta^2-c^2)}{\left(\left(n-\frac{1}{2}\right)^2\pi^2+\theta^2\right)^2} < 0 \qquad \text{for} \quad \theta > c\,,$$

$$\begin{aligned}
V\left(\frac{dY_1(\theta)}{d\theta}\right) &= \sum_{n=1}^{\infty}\frac{2\theta^2\left(\left(n-\frac{1}{2}\right)^2\pi^2+c^2\right)^2}{\left(\left(n-\frac{1}{2}\right)^2\pi^2+\theta^2\right)^4} \\
&\le \sum_{n=1}^{\infty}\frac{2\theta^2}{\left(\left(n-\frac{1}{2}\right)^2\pi^2+\theta^2\right)^2} \\
&= \frac{1}{2\theta}\left(\tanh\theta-\theta\,\mathrm{sech}^2\theta\right) = O\left(\frac{1}{\theta}\right).
\end{aligned}$$

Then, using the same reasoning as before, we can ensure the existence of the local maximum $\hat{\kappa}$ such that $\hat{\kappa} = O_p(T^{-1})$.

7.3 We first obtain

$$E\left[\exp\left\{i\theta\sum_{n=1}^{\infty}\frac{Z_n^2}{\left(n-\frac{1}{2}\right)^2\pi^2}\right\}\right] = \left(\cos\sqrt{2i\theta}\right)^{-\frac{1}{2}}.$$

Then (8.104) follows from Theorem 5.13.

7.4 It is easy to deduce that

$$\begin{aligned}
\psi_1(\theta) &= \prod_{n=1}^{\infty}\left[1-\frac{2i\theta\left(\left(n-\frac{1}{2}\right)^2\pi^2+c^2\right)}{\left(\left(n-\frac{1}{2}\right)^2\pi^2+x^2\right)^2}\right]^{-\frac{1}{2}} \\
&= \prod_{n=1}^{\infty}\left[\frac{\left(1-\frac{a(\theta)+b(\theta)}{\left(n-\frac{1}{2}\right)^2\pi^2}\right)\left(1-\frac{a(\theta)-b(\theta)}{\left(n-\frac{1}{2}\right)^2\pi^2}\right)}{\left(1+\frac{x^2}{\left(n-\frac{1}{2}\right)^2\pi^2}\right)^2}\right]^{-\frac{1}{2}}
\end{aligned}$$

$$= \cosh x \left[\cos \sqrt{a(\theta)+b(\theta)}\, \cos \sqrt{a(\theta)-b(\theta)}\right]^{-\frac{1}{2}} .$$

7.5 Noting that $\Delta y \sim \mathrm{N}(0, \sigma_\varepsilon^2(\Omega + \rho_0 I_{T-1}))$ we have

$$\begin{aligned}
&\mathcal{L}\left(\frac{1}{\sigma_\varepsilon^2}\Delta y' \left((\Omega + \rho I_{T-1})^{-1} - \Omega^{-1}\right)\Delta y\right) \\
&= \mathcal{L}\left(\sum_{j=1}^{T-1}\left(\rho_0 + 4s_{jT}^2\right)\left(\frac{1}{\rho + 4s_{jT}^2} - \frac{1}{4s_{jT}^2}\right)Z_j^2\right) \\
&= \mathcal{L}\left(-\theta^2 \sum_{j=1}^{T-1} \frac{c^2 + 4T^2 s_{jT}^2}{4T^2 s_{jT}^2\left(\theta^2 + 4T^2 s_{jT}^2\right)} Z_j^2\right) ,
\end{aligned}$$

where $s_{jT} = \sin\left(j\pi/\left(2T\right)\right)$. Using the same arguments as in the solution to Problem 2.2 we can establish (8.109).

7.6 Let us consider $|\,\Omega + \rho I_{T-1}\,| = (\rho+2)^{T-1} D_T$, where $D_T = D_{T-1} - a^2 D_{T-2}$ with $a = -1/(\rho+2)$. When $\rho \neq 0$, we have $D_T = c_1 x_1^{T-1} + c_2 x_2^{T-1}$, where $D_1 = 1,\ D_2 = 1 - a^2$ and

$$\begin{aligned}
x_1 &= \frac{\rho + 2 + \sqrt{\rho^2 + 4\rho}}{2(\rho+2)} = \frac{1}{\rho+2}\left(1 + \frac{\theta}{T} + O\left(\frac{1}{T^2}\right)\right) , \\
x_2 &= \frac{\rho + 2 - \sqrt{\rho^2 + 4\rho}}{2(\rho+2)} = \frac{1}{\rho+2}\left(1 - \frac{\theta}{T} + O\left(\frac{1}{T^2}\right)\right) , \\
c_1 &= \frac{\rho^2 + 4\rho + 2 + (\rho+2)\sqrt{\rho^2 + 4\rho}}{2(\rho+2)\sqrt{\rho^2+4\rho}} = \frac{T}{2\theta}\left(1 + O\left(\frac{1}{T}\right)\right) , \\
c_2 &= \frac{-\rho^2 - 4\rho - 2 + (\rho+2)\sqrt{\rho^2 + 4\rho}}{2(\rho+2)\sqrt{\rho^2+4\rho}} = -\frac{T}{2\theta}\left(1 + O\left(\frac{1}{T}\right)\right) .
\end{aligned}$$

Then (8.110) is established by noting that $|\,\Omega\,| = T$.

Chapter 9.

2.1 The (j,k)-th element $\Phi_{jk}(\rho)$ of $\Phi(\rho)$ is given for $j \leq k$ by

$$\Phi_{jk}(\rho) = \sum_{i=0}^{j-1} \rho^{k-j+2i} .$$

Therefore we obtain, for $j \leq k$,

$$\text{(S14)} \qquad \frac{d\Phi_{jk}(\rho)}{d\rho} = \sum_{i=0}^{j-1} (k - j + 2i)\, \rho^{k-j+2i-1}$$

so that $d\Phi_{jk}(\rho)/\,d\rho\big|_{\rho=1} = jk - j = jk - \min(j,k)$. This yields (9.12).

2.2 Under H_0 it holds that $S_{T1}/T = X/(X+Y)$, where

$$\begin{aligned} X &= \frac{1}{T\sigma^2}y_T^2 = \left(\frac{1}{\sqrt{T}\sigma}\sum_{j=1}^{T}\varepsilon_j\right)^2 \sim \chi^2(1)\,, \\ Y &= \frac{1}{\sigma^2}\sum_{j=1}^{T}\left(\varepsilon_j - \frac{1}{T}\sum_{i=1}^{T}\varepsilon_i\right)^2 \sim \chi^2(T-1)\,. \end{aligned}$$

The conclusion follows from X and Y being independent.

2.3 It is easy to obtain $Q_2'\Phi(1)Q_2 = I_{T-1}$ and $Q_2Q_2' = (C^{-1}(1))'\,C^{-1}(1) - e_1e_1'$, where $e_1 = (1,0,\cdots,0)' : T\times 1$. Thus $y'Q_2Q_2'y = \sum_{j=2}^{T}(y_j - y_{j-1})^2$ and $y'Q_2Q_2'dd'Q_2Q_2'y = (y_T - y_1)^2$.

2.4 Under H_0 it holds that $S_{T2}/\,T = X/\,(X+Y)$, where

$$\begin{aligned} X &= \frac{1}{(T-1)\sigma^2}(y_T - y_1)^2 = \left(\frac{1}{\sqrt{T-1}\sigma}\sum_{j=2}^{T}\varepsilon_j\right)^2 \sim \chi^2(1)\,, \\ Y &= \frac{1}{\sigma^2}\sum_{j=2}^{T}\left(\varepsilon_j - \frac{1}{T-1}\sum_{i=2}^{T}\varepsilon_i\right)^2 \sim \chi^2(T-2)\,. \end{aligned}$$

The conclusion follows from the independence of X and Y.

2.5 It follows from (S14) that, for $j \le k$,

$$\begin{aligned} \left.\frac{d^2\Phi_{jk}(\rho)}{d\rho^2}\right|_{\rho=1} &= \sum_{i=0}^{j-1}(k-j+2i)(k-j+2i-1) \\ &= jk(k-3) + \frac{j(j^2+5)}{3}\,, \end{aligned}$$

which yields (9.18).

2.6 It can be shown that

$$(Q_3'\Phi(1)Q_3)^{-1}Q_3' = (-I_{T-1},\,0) + \frac{1}{T}(0,\cdots,0,\tilde{d})\,,$$

where $\tilde{d} = (1,\cdots,T-1)'$. This yields the last expression for S_{T3}.

2.7 Let us put $Z = (Q_3'\Phi(1)Q_3)^{-\frac{1}{2}}Q_3'y/\sigma$ so that

$$S_{T3} = \frac{1}{T}\frac{Z'(Q_3'\Phi(1)Q_3)^{-1}Z}{Z'Z}\,.$$

Since $Z \sim \mathrm{N}(0,\, I_{T-1})$ under H_0, $Z'Z/T$ converges in probability to unity. Noting that the (j,k)-th element of $(Q_3'\Phi(1)Q_3)^{-1}$ is $\min(j,k)-jk/T$, we can establish (9.23) from Theorem 5.12.

2.8 It can be shown after some algebra that

$$\begin{aligned}(Q_4'\Phi(1)Q_4)^{-1}Q_4' \quad &= \begin{pmatrix} 0 & -1 & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & -1 & 0 \end{pmatrix} \\ &\quad + \frac{1}{T-1}\begin{pmatrix} T-2 & & \\ \vdots & & 0 \\ 1 & & \end{pmatrix} + \frac{1}{T-1}\begin{pmatrix} & & 1 \\ 0 & & \vdots \\ & & T-2 \end{pmatrix},\end{aligned}$$

which yields the last expression for S_{T4}.

2.9 Let us put $Z = (Q_4'\Phi(1)Q_4)^{-\frac{1}{2}}Q_4'y/\sigma$ so that

$$S_{T4} = \frac{1}{T}\frac{Z'(Q_4'\Phi(1)Q_4)^{-1}Z}{Z'Z}\,.$$

Noting that the (j,k)-th element of $(Q_4'\Phi(1)Q_4)^{-1}$ is $\min(j,k) - jk/(T-1)$, we can deduce that the limiting null distribution of S_{T4} is the same as that of S_{T3} given in (9.23).

3.1 It can be easily checked that Lemma 9.1 holds if Q is replaced by XG, where G is a $p \times p$ nonsingular matrix and $P'X = 0$. Then (9.34) follows from (9.36) by putting $P = H$, $A = \Sigma(\theta_0)$ and $Q = XG$. Noting that $H(H'\Sigma(\theta_0)H)^{-1}H' = \tilde{M}'\Sigma^{-1}(\theta_0)\tilde{M} = \tilde{M}'\Sigma^{-1}(\theta_0) = \Sigma^{-1}(\theta_0)\tilde{M}$ we can also prove (9.35).

3.2 The first equality comes from the fact that $d\Phi^{-1}(\rho)/d\rho = -\Phi^{-1}(\rho)d\Phi(\rho)/d\rho\Phi^{-1}(\rho)$. Since $d\Phi^{-1}(\rho)\big/\, d\rho\Big|_{\rho=1} = \Phi^{-1}(1) - e_Te_T'$, we have

$$LM_1' = -\frac{\tilde{\eta}'\left(\Phi^{-1}(1) - e_Te_T'\right)\tilde{\eta}}{\tilde{\eta}'\Phi^{-1}(1)\tilde{\eta}}\,,$$

which yields the second equality.

3.3 We first have $\tilde{\eta}_T = e_T' \tilde{M} y$, where it holds that $X'\Phi^{-1}(1)\tilde{M} = 0$. Thus $\tilde{\eta}_T$ reduces to 0 if e_T belongs to the column space of $\Phi^{-1}(1)X$ or if $\Phi(1)e_T = d$ belongs to the column space of X.

3.4 As for Model C we have

$$\begin{aligned}
\tilde{\eta} &= \left[I_T - d\left(d'\Phi^{-1}(1)d\right)^{-1} d'\Phi^{-1}(1)\right] y \\
&= \left(I_T - \frac{1}{T} d e_T' \right) y = y - \frac{1}{T} y_T d \,,
\end{aligned}$$

which gives S_{T3} in (9.22). For Model D we have

$$\begin{aligned}
\tilde{\eta} &= \left[I_T - (e,d) \left(\begin{pmatrix} e' \\ d' \end{pmatrix} \Phi^{-1}(1)(e,d) \right)^{-1} \begin{pmatrix} e' \\ d' \end{pmatrix} \Phi^{-1}(1) \right] y \\
&= \left[I_T - \frac{1}{T-1}(Tee_1' - de_1' - ee_T' + de_T') \right] y \\
&= y - y_1 e - \frac{1}{T-1}(y_T - y_1)(d - e) \,,
\end{aligned}$$

which gives S_{T4} in (9.26).

5.1 We have already proved (9.46) in Section 2 of Chapter 7. As for (9.47) it can be shown that $\tilde{\eta}_1 = O_p(1)$ for all models. For $j \geq 2$ we have

$$\tilde{\eta}_j - \tilde{\eta}_{j-1} = \begin{cases} \eta_j - \eta_{j-1} \,, & \text{Models A, B,} \\[2ex] \eta_j - \eta_{j-1} - \dfrac{1}{T}\eta_T \,, & \text{Model C,} \\[2ex] \eta_j - \eta_{j-1} - \dfrac{1}{T-1}(\eta_T - \eta_1) \,, & \text{Model D.} \end{cases}$$

Noting that $\eta_j - \eta_{j-1} = -c\eta_{j-1}/T + \varepsilon_j$ and $\eta_T = O_p(\sqrt{T})$, we can establish (9.47).

5.2 Let us first consider

$$\frac{1}{T\sigma^2}\tilde{\eta}_T^2 = \frac{1}{T\sigma^2} \sum_{j,k=1}^{T} \left(1 - \frac{c}{T}\right)^{2T-j-k} \varepsilon_j \, \varepsilon_k \,,$$

which converges in distribution to

$$\int_0^1 \int_0^1 e^{-c(2-s-t)} dw(s)dw(t) \sim \frac{1-e^{-2c}}{2c}\chi^2(1)\,.$$

Consider next

$$\begin{aligned}\frac{1}{T^2\sigma^2}\sum_{j=1}^T (\tilde{\eta}_j - \bar{\tilde{\eta}})^2 &= \frac{1}{T^2\sigma^2}\sum_{j=1}^T (\eta_j - \bar{\eta})^2 \\ &= \frac{1}{T}\sum_{j=1}^T \left(Y_T\left(\frac{j}{T}\right) - \frac{1}{T}\sum_{k=1}^T Y_T\left(\frac{k}{T}\right)\right)^2,\end{aligned}$$

where

$$\text{(S15)} \qquad Y_T(t) = \frac{1}{\sqrt{T}\sigma}\eta_{j-1} + T\left(t - \frac{j-1}{T}\right)\frac{\eta_j - \eta_{j-1}}{\sqrt{T}\sigma}\,, \quad \left(\frac{j-1}{T} \le t \le \frac{j}{T}\right).$$

Since $\mathcal{L}(Y_T) \to \mathcal{L}(Y)$ by the FCLT, the weak convergence result on R_3 follows from the continuous mapping theorem.

5.3 Let us define $dZ(t) = -\gamma Z(t)dt + dw(t)$ with $Z(0) = 0$. Then Girsanov's theorem yields

$$\begin{aligned}E\left[\exp\left\{\theta\int_0^1 Y^2(t)dt\right\}\right] &= E\left[\exp\left\{\frac{\gamma - c}{2}\left(Z^2(1) - 1\right)\right\}\right] \\ &= e^{c/2}\left[\cosh\gamma + c\,\frac{\sinh\gamma}{\gamma}\right]^{-\frac{1}{2}},\end{aligned}$$

where $\gamma = \sqrt{c^2 - 2\theta}$. This gives us the expression for $\beta_2(\alpha)$. The expression for $\beta_3(\alpha)$ can be proved similarly.

5.4 It is easy to see that $2c/(1-e^{-2c}) \to \infty$ as $c \to \infty$ so that $\beta_1(\alpha) \to 1$. In Section 5 of Chapter 7 we have proved that, as $c \to \infty$,

$$c\int_0^1 Y^2(t)dt \longrightarrow \frac{1}{2}, \qquad c\int_0^1 \left(Y(t) - \int_0^1 Y(s)ds\right)^2 dt \longrightarrow \frac{1}{2},$$

$$\int_0^1 Y(t)dY(t) \Big/ \left(c\int_0^1 Y^2(t)dt\right) \longrightarrow -1\,,$$

in probability. The above facts imply that $\beta_k(\alpha) \to 1$ as $c \to \infty$ for $k = 2, \cdots, 6$.

5.5 Noting that $\hat{\eta}_j = \eta_j - \bar{\eta}$ we have

$$\frac{1}{T^2\sigma^2}\sum_{j=1}^{T}\hat{\eta}_j^2 = \frac{1}{T}\sum_{j=1}^{T}\left(Y_T\left(\frac{j}{T}\right) - \frac{1}{T}\sum_{k=1}^{T}Y_T\left(\frac{k}{T}\right)\right)^2 ,$$

$$\frac{1}{T\sigma^2}\sum_{j=1}^{T}(\hat{\eta}_j - \hat{\eta}_{j-1})^2 = \left(\frac{1}{T}\sum_{j=1}^{T}Y_T\left(\frac{j}{T}\right)\right)^2 + \frac{1}{T\sigma^2}\sum_{j=2}^{T}(\eta_j - \eta_{j-1})^2 + o_p(1) ,$$

where $Y_T(t)$ is defined in (S15). Then the weak convergence result on R_6 follows from the FCLT and the continuous mapping theorem.

5.6 Defining $dZ(t) = -\gamma Z(t)dt + dw(t)$ with $Z(0) = 0$ we consider

$$\begin{aligned} & E\left[\exp\left\{\theta x + \theta x\left(\int_0^1 Y(t)dt\right)^2 - \theta\int_0^1\left(Y(t) - \int_0^1 Y(s)ds\right)^2 dt\right\}\right] \\ &= e^{\theta x}E\left[\exp\left\{\frac{\gamma - c}{2}\left(Z^2(1) - 1\right) + \theta(x+1)\left(\int_0^1 Z(t)dt\right)^2\right\}\right] , \end{aligned}$$

where $\gamma = \sqrt{c^2 + 2\theta}$. This leads us to the expressions for $\psi_1(\theta; x)$ and $\beta_6(\alpha)$.

5.7 Noting that $\tilde{\eta}_j = \eta_j - j\eta_T/T$ we have

$$\begin{aligned} \frac{1}{T^2\sigma^2}\sum_{j=1}^{T}(\tilde{\eta}_j - \bar{\tilde{\eta}})^2 &= \frac{1}{T^2\sigma^2}\sum_{j=1}^{T}\left\{\eta_j - \frac{j}{T}\eta_T - \left(\bar{\eta} - \frac{T+1}{2T}\eta_T\right)\right\}^2 \\ &= \frac{1}{T}\sum_{j=1}^{T}\left\{Y_T\left(\frac{j}{T}\right) - \frac{1}{T}\sum_{k=1}^{T}Y_T\left(\frac{k}{T}\right)\right. \\ &\qquad \left. - \left(\frac{j}{T} - \frac{1}{2}\right)Y_T(1)\right\}^2 + o_p(1) , \end{aligned}$$

where $Y_T(t)$ is defined in (S15). Then the weak convergence result on R_3 is easily established.

5.8 Defining $dZ(t) = -\gamma Z(t)dt + dw(t)$ with $Z(0) = 0$ we consider

$$\begin{aligned} & E\left[\exp\left\{\theta\int_0^1\left(Y(t) - \int_0^1 Y(s)ds - \left(t - \frac{1}{2}\right)Y(1)\right)^2 dt\right\}\right] \\ &= E\left[\exp\left\{\frac{\gamma - c}{2}\left(Z^2(1) - 1\right) - \theta\left(\int_0^1 Z(t)dt\right)^2\right.\right. \\ &\qquad \left.\left. -2\theta Z(1)\int_0^1\left(t - \frac{1}{2}\right)Z(t)dt + \frac{\theta}{12}Z^2(1)\right\}\right] , \end{aligned}$$

where $\gamma = \sqrt{c^2 - 2\theta}$. This yields $\phi_3(-i\theta)$ after some algebra.

5.9 Noting that

$$\hat{\eta}_j = \eta_j + \left(\frac{6j}{T} - 4\right)\frac{1}{T}\sum_{j=1}^{T}\eta_j - \left(\frac{12j}{T} - 6\right)\frac{1}{T^2}\sum_{j=1}^{T} j\eta_j + o_p(1)\,,$$

we have

$$\hat{\eta}_1 = -\frac{4}{T}\sum_{j=1}^{T}\eta_j + \frac{6}{T^2}\sum_{j=1}^{T} j\eta_j + O_p(1)\,,$$

$$\hat{\eta}_j - \hat{\eta}_{j-1} = \eta_j - \eta_{j-1} + o_p(1)\,, \qquad (j \geq 2)\,.$$

The weak convergence result on R_6 follows from the above relations.

5.10 We consider

$$\begin{aligned} & E\left[\exp\left\{\theta x + \theta x\left(4\int_0^1 Y(t)dt - 6\int_0^1 tY(t)dt\right)^2 - \theta V_4\right\}\right] \\ & = e^{\theta x} E\left[\exp\left\{\frac{\gamma - c}{2}\left(Z^2(1) - 1\right) + 4\theta(4x+1)\left(\int_0^1 Z(t)dt\right)^2\right.\right. \\ & \qquad \left.\left. + 12\theta(3x+1)\left(\int_0^1 tZ(t)dt\right)^2 - 12\theta(4x+1)\int_0^1 Z(t)dt\int_0^1 tZ(t)dt\right\}\right]\,, \end{aligned}$$

where $dZ(t) = -\gamma Z(t)dt + dw(t)$ with $Z(0) = 0$ and $\gamma = \sqrt{c^2 + 2\theta}$. This yields $e^{\theta x}\psi_2(-i\theta; x)$ after some algebra; hence we obtain the expression for $\beta_6(\alpha)$.

6.1 Let $f(v|\rho)$ be the density of the maximal invariant $v = H'y/\sqrt{y'HH'y}$, where $f(v|\rho)$ is defined as in (9.10). Then the Neyman-Pearson lemma ensures that the test which rejects H_0 for large values of $f(v|1 - (\theta/T))/f(v|1)$ is MPI. By using Lemma 9.1, this is seen to be equivalent to rejecting H_0 when $V_T^{(M)}(\theta)$ in (9.51) takes large values.

6.2 The weak convergence result on $V_T^{(A)}(\theta)$ is proved in the text. Consider $V_T^{(B)}(\theta)$ in (9.51), where $\tilde{\eta}_j^{(0)} = \eta_j - \eta_1$ and

$$\begin{aligned} \tilde{\eta}_j^{(1)} &= \eta_j - \frac{1}{1 + (T-1)(1-\rho)^2} \\ & \quad \times \left[\left(1 - \rho + \rho^2\right)\eta_1 + (1-\rho)^2\left(\eta_2 + \cdots + \eta_{T-1}\right) + (1-\rho)\,\eta_T\right] \end{aligned}$$

with $\rho = 1 - (\theta/T)$. Then it is easy to deduce that $\mathcal{L}\left(V_T^{(B)}(\theta)\right) \longrightarrow \mathcal{L}\left(V^{(B)}(c,\theta)\right) = \mathcal{L}\left(V^{(A)}(c,\theta)\right)$. For Model C we have $\tilde{\eta}_j^{(0)} = \eta_j - j\eta_T \big/ T$ so that the denominator of $V_T^{(C)}(\theta)$ divided by T converges in probability to σ^2. Since

$$\begin{aligned}
\tilde{\eta}_j^{(1)} &= y_j - j\tilde{\beta}^{(1)} = \eta_j - \frac{j}{\delta T + O(1)} \sum_{j=1}^{T} (1 + (1-\rho)j)(\eta_j - \rho\eta_{j-1}) \\
&= \eta_j - \frac{j}{\sqrt{T}} A_T \,,
\end{aligned}$$

the numerator of $V_T^{(C)}(\theta)$ is

$$\begin{aligned}
&\sum_{j=1}^{T} \left(\eta_j - \eta_{j-1} - \frac{1}{T}\eta_T\right)^2 - \sum_{j=1}^{T} \left(\tilde{\eta}_j^{(1)} - \tilde{\eta}_{j-1}^{(1)} + (1-\rho)\tilde{\eta}_{j-1}^{(1)}\right)^2 \\
&= -\frac{1}{T}\eta_T^2 + \frac{2}{\sqrt{T}} A_T \eta_T - A_T^2 - \theta\left(\frac{1}{\sqrt{T}}\eta_T - A_T\right)^2 \\
&\quad -\frac{\theta^2}{T} \sum_{j=1}^{T} \left(\frac{1}{\sqrt{T}}\eta_j - \frac{j}{T} A_T\right)^2 + \theta + o_p(1) \,.
\end{aligned}$$

The joint weak convergence and the fact that

$$\mathcal{L}(A_T) \longrightarrow \mathcal{L}\left(\frac{\sigma}{\delta}\left((\theta+1)Y(1) + \theta^2 \int_0^1 tY(t)dt\right)\right)$$

lead us to deduce that $\mathcal{L}\left(V_T^{(C)}(\theta)\right) \longrightarrow \mathcal{L}\left(V^{(C)}(c,\theta)\right)$. For Model D it can be checked that

$$\begin{aligned}
\tilde{\eta}_j^{(0)} - \tilde{\eta}_{j-1}^{(0)} &= \eta_j - \eta_{j-1} - \frac{1}{T-1}(\eta_T - \eta_1) \,, \\
\tilde{\eta}_j^{(1)} - \tilde{\eta}_{j-1}^{(1)} &= \eta_j - \eta_{j-1} - \frac{1}{\sqrt{T}} A_T + O_p\left(\frac{1}{T}\right) \,.
\end{aligned}$$

Thus it holds that $\mathcal{L}\left(V_T^{(D)}(\theta)\right) \longrightarrow \mathcal{L}\left(V^{(D)}(c,\theta)\right) = \mathcal{L}\left(V^{(C)}(c,\theta)\right)$.

6.3 For Models A and B we easily obtain, by Girsanov's theorem,

$$\begin{aligned}
E\left[\exp\left\{u\left(c - V^{(A)}(c,c)\right)\big/ c^2\right\}\right] &= E\left[\exp\left\{u\int_0^1 Y^2(t)dt + \frac{u}{c}Y^2(1)\right\}\right] \\
&= \left[\left(\cos\mu - \frac{\mu}{c}\sin\mu\right)\Big/ e^c\right]^{-\frac{1}{2}} \,, \\
E\left[\exp\left\{u\left(c - V^{(A)}(0,c)\right)\big/ c^2\right\}\right] &= E\left[\exp\left\{u\int_0^1 w^2(t)dt + \frac{u}{c}w^2(1)\right\}\right] \\
&= \left[\cos\nu - \frac{\nu}{c}\sin\nu\right]^{-\frac{1}{2}} \,.
\end{aligned}$$

We can compute $E\left[\exp\left\{u\left(c-V^{(C)}(c,c)\right)\big/ c^2\right\}\right]$ and $E\left[\exp\left\{u\left(c-V^{(C)}(0,c)\right)\big/ c^2\right\}\right]$ similarly, which establishes the theorem.

6.4 For Models C and D we obtain, by Girsanov's theorem,

$$
\begin{aligned}
&E\left[\exp\left\{u\left(\theta-V^{(C)}(c,\theta)\right)\big/\theta^2\right\}\right] \\
&= E\left[\exp\left\{u\left(\int_0^1 Y^2(t)dt+\frac{\theta+1}{3\delta}Y^2(1)\right.\right.\right. \\
&\qquad\left.\left.\left.-\frac{2(\theta+1)}{\delta}Y(1)\int_0^1 tY(t)dt-\frac{\theta^2}{\delta}\left(\int_0^1 tY(t)dt\right)^2\right)\right\}\right] \\
&= \exp\left(\frac{c-\beta}{2}\right)E\left[\exp\left\{\left(\frac{\beta-c}{2}+\frac{u(\theta+1)}{3\delta}\right)Z^2(1)-\frac{u\theta^2}{\delta}\left(\int_0^1 tZ(t)dt\right)^2\right.\right. \\
&\qquad\left.\left.-\frac{2u(\theta+1)}{\delta}Z(1)\int_0^1 tZ(t)dt\right\}\right],
\end{aligned}
$$

where $dZ(t)=-\beta Z(t)dt+dw(t)$ with $Z(0)=0$ and $\beta=\sqrt{c^2-2u}$. We can arrive, after some algebra, at $\phi^{(C)}(-iu;c,\theta)$. We can compute $E\left[\exp\left\{u\left(\theta-V^{(A)}(c,\theta)\right)\big/\theta^2\right\}\right]$ similarly, which establishes the theorem.

7.1 Let us consider $T\left(\hat\rho(\delta)-1\right)=U_T/V_T$, where

$$
\begin{aligned}
U_T &= \frac{1}{T\sigma^2}\left[\sum_{j=2}^{T} y_{j-1}\left(y_j-y_{j-1}\right)-\delta y_T^2\right] \\
&= \frac{1}{2}\left(Y_T^2(1)-1\right)-\delta Y_T^2(1)+o_p(1), \\
V_T &= \frac{1}{T^2\sigma^2}\left[\sum_{j=2}^{T} y_{j-1}^2+\delta y_T^2\right] \\
&= \frac{1}{T}\sum_{j=1}^{T} Y_T^2\left(\frac{j}{T}\right)+o_p(1),
\end{aligned}
$$

with $Y_T(t)$ defined in (S15). Then we can establish the first equality in (9.59). The second equality can also be proved by using Girsanov's theorem.

7.2 Let us put $x_j=(y_{j-1},\, y_{j-2})'$ and $G=(G_1,\, G_2)$, where $G_1=(1,\,-\rho)'$ and

$G_2 = (1,\, 0)'$. Then $G'x_j = (\varepsilon_{j-1},\, y_{j-1})'$ and

$$
\begin{aligned}
\begin{pmatrix} \hat{\rho}_1 - \rho \\ \\ \hat{\rho}_2 \end{pmatrix} &= \left(\Sigma x_j x_j'\right)^{-1} \Sigma x_j\, \varepsilon_j \\
&= G \begin{pmatrix} \Sigma \varepsilon_{j-1}^2 & \Sigma \varepsilon_{j-1}\, y_{j-1} \\ \\ \Sigma \varepsilon_{j-1}\, y_{j-1} & \Sigma y_{j-1}^2 \end{pmatrix}^{-1} \begin{pmatrix} \Sigma \varepsilon_{j-1}\, \varepsilon_j \\ \\ \Sigma y_{j-1}\, \varepsilon_j \end{pmatrix}.
\end{aligned}
$$

Thus we obtain $\sqrt{T}\,(\hat{\rho}_1 - \rho) = A/B$, where

$$
\begin{aligned}
A &= \left(\frac{1}{T}\sum y_{j-1}^2 - \frac{1}{T}\sum \varepsilon_{j-1} y_{j-1}\right) \frac{1}{\sqrt{T}} \sum \varepsilon_{j-1}\varepsilon_j \\
&\qquad + \left(\frac{1}{T}\sum \varepsilon_{j-1}^2 - \frac{1}{T}\sum \varepsilon_{j-1} y_{j-1}\right) \frac{1}{\sqrt{T}} \sum y_{j-1}\varepsilon_j \\
&= \frac{\rho^2\sigma^2}{1-\rho^2} \frac{1}{\sqrt{T}} \sum \varepsilon_{j-1}\varepsilon_j + o_p(1)\,, \\
B &= \frac{1}{T}\sum \varepsilon_{j-1}^2 \frac{1}{T}\sum y_{j-1}^2 - \left(\frac{1}{T}\sum \varepsilon_{j-1} y_{j-1}\right)^2 \\
&= \frac{\sigma^4\rho^2}{1-\rho^2} + o_p(1)\,.
\end{aligned}
$$

Then we can deduce that $\sqrt{T}\,(\hat{\rho}_1 - \rho) \longrightarrow \mathrm{N}\,(0, 1)$ so that

$$
P\left(\sqrt{T}\,(\hat{\rho}_1 - 1) \le x\right) = P\left(\sqrt{T}\,(\hat{\rho}_1 - \rho) \le x + \sqrt{T}(1-\rho)\right) \cong \Phi\left(x + \sqrt{T}(1-\rho)\right).
$$

8.1 The LBI test rejects H_0 when

$$
S_T = -\frac{\tilde{\eta}' \left(\left.\dfrac{d\Phi^{-1}(\rho_m)}{d\rho_m}\right|_{\rho_m = 1} \otimes I_m \right) \tilde{\eta}}{\tilde{\eta}' \left(\Phi^{-1}(1) \otimes I_m\right) \tilde{\eta}} < c\,.
$$

Since $d\Phi^{-1}(\rho)/\, d\rho|_{\rho=1} = \Phi^{-1}(1) - e_N e_N'$ with $e_N = (0, \cdots, 0, 1)' : N \times 1$, the above test is seen to be equivalent to the one given in Theorem 9.17.

8.2 Let us put

$$
\sum_{j=1}^{m} \tilde{\eta}_{T-m+j}^2 = a_N' a_N\,,
$$

where $a_N = (e_N' \otimes I_m)\left(\tilde{M} \otimes I_m\right) y = \left(e_N'\tilde{M} \otimes I_m\right) y$ with $e_N = (0, \cdots, 0, 1)' : N \times 1$. Since $\bar{X}'\Phi^{-1}(1)\tilde{M} = 0$, a_N reduces to 0 if e_N belongs to the column space of $\Phi^{-1}(1)\bar{X}$ or if $\Phi(1)e_N = d$ belongs to the column space of $\bar{X}$.

8.3 The LBIU test rejects H_0 when

$$S_T = -\frac{\tilde{\eta}'\left(\left.\dfrac{d^2\Phi^{-1}(\rho_m)}{d\rho_m^2}\right|_{\rho_m=1} \otimes I_m\right)\tilde{\eta}}{\tilde{\eta}'\left(\Phi^{-1}(1) \otimes I_m\right)\tilde{\eta}} > c\,.$$

Since $\left. d^2\Phi^{-1}(\rho)/\,d\rho^2\right|_{\rho=1} = 2(I_N - e_N e_N')$ and $\tilde{\eta}'(e_N e_N' \otimes I_m)\tilde{\eta} = 0$, the above test is seen to be equivalent to the one given in Theorem 9.18.

8.4 Since it can be shown that

$$\begin{aligned}
\tilde{\eta} &= \left[I_T - (d \otimes I_m)\left(d'\Phi^{-1}(1)d \otimes I_m\right)^{-1}\left(d'\Phi^{-1}(1) \otimes I_m\right)\right] y \\
&= \left(I_T - \frac{1}{N}(d \otimes I_m)(e_N' \otimes I_m)\right) y \\
&= y - \frac{1}{N}(d \otimes I_m)\, \underset{\sim}{y}_N\,,
\end{aligned}$$

we can obtain the rejection region (9.71) from Theorem 9.18.

8.5 Putting $\bar{X} = (e, d)$ we can show that

$$\begin{aligned}
\tilde{\eta} &= \left[I_T - \left(\bar{X} \otimes I_m\right)\left(\bar{X}'\Phi^{-1}(1)\bar{X} \otimes I_m\right)^{-1}\left(\bar{X}'\Phi^{-1}(1) \otimes I_m\right)\right] y \\
&= \left[I_T - ee_1' \otimes I_m - \frac{1}{N-1}((d-e)(e_N' - e_1')) \otimes I_m\right] y \\
&= y - (e \otimes I_m)\, \underset{\sim}{y}_1 - \frac{1}{N-1}((d-e) \otimes I_m)\left(\underset{\sim}{y}_N - \underset{\sim}{y}_1\right),
\end{aligned}$$

which yields the rejection region (9.72).

8.6 We first note that R_{C2} may be rewritten as

$$\begin{aligned}
R_{C2} &= \frac{m}{N}\,\frac{\tilde{\eta}'\tilde{\eta}}{\tilde{\eta}'\left(\Phi^{-1}(1) \otimes I_m\right)\tilde{\eta}} \\
&= \frac{m}{N}\,\frac{\varepsilon'\left(B_N \otimes I_m\right)\varepsilon}{\varepsilon'\left(A_N \otimes I_m\right)\varepsilon}\,,
\end{aligned}$$

where $A_N = C'(\rho_m)\tilde{M}'\Phi^{-1}(1)\tilde{M}C(\rho_m)$ and $B_N = C'(\rho_m)\tilde{M}'\tilde{M}C(\rho_m)$ with $\tilde{M} = I_N - d(d'\Phi^{-1}(1)d)^{-1}d'\Phi^{-1}(1)$. We have already shown that, when $m = 1$,

$$\frac{1}{N\sigma^2}\,\varepsilon' A_N \varepsilon \longrightarrow 1 \qquad \text{in probability}$$

so that, for general m,

$$\frac{1}{N\sigma^2}\,\varepsilon'\left(A_N \otimes I_m\right)\varepsilon \longrightarrow m \qquad \text{in probability.}$$

We also have

$$\mathcal{L}\left(\frac{1}{N^2\sigma^2}\,\varepsilon'\left(B_N \otimes I_m\right)\varepsilon\right) \longrightarrow \mathcal{L}\left(\int_0^1\int_0^1 K(s,t)\,d\underset{\sim}{w}'(s)\,d\underset{\sim}{w}(t)\right),$$

where $K(s,t)$ is a positive definite kernel and $\left\{\underset{\sim}{w}(t)\right\}$ is the m-dimensional standard Brownian motion. Since the c.f. of this last limiting distribution is given by $\left(\phi_2(\theta)\right)^m$, we obtain the conclusion.

9.1 The statistic R_2 may be rewritten as

$$R_2 = \frac{1}{T^2}u'B_T\,u \bigg/ \frac{1}{T}u'A_T\,u\,,$$

where $A_T = C'(\rho)\tilde{M}'\Phi^{-1}(1)\tilde{M}C(\rho)$ and $B_T = C'(\rho)\tilde{M}'\tilde{M}C(\rho)$. We have shown that

$$\frac{1}{T\sigma^2}\,\varepsilon' A_T\,\varepsilon \longrightarrow 1 \qquad \text{in probability,} \qquad \mathcal{L}\left(\frac{1}{T^2\sigma^2}\,\varepsilon' B_T\,\varepsilon\right) \longrightarrow \mathcal{L}(W_2)\,.$$

Then $u'A_T\,u/T \to \sigma_S^2$ in probability and it follows from (5.161) that

$$\mathcal{L}\left(\frac{1}{T^2}\,u'B_T\,u\right) \longrightarrow \mathcal{L}(\sigma_L^2\,W_2)$$

so that $\mathcal{L}(R_2) \to \mathcal{L}(W_2/r)$. The weak convergence result on R_6 can be proved similarly.

9.2 The weak convergence results for Models A and C are obvious. As for Models B and D let us consider

$$\frac{1}{R_6} + \frac{\tilde{\sigma}_L^2 - \tilde{\sigma}_S^2}{\sum_{j=1}^{T}\hat{\eta}_j^2 \Big/ T^2} = \left(\frac{1}{T}\sum_{j=1}^{T}\left(\hat{\eta}_j - \hat{\eta}_{j-1}\right)^2 + \tilde{\sigma}_L^2 - \tilde{\sigma}_S^2\right) \bigg/ \frac{1}{T^2}\sum_{j=1}^{T}\hat{\eta}_j^2\,.$$

This converges in distribution to

$$\left(\sigma_L^2 X_6 + \sigma_S^2 + \sigma_L^2 - \sigma_S^2\right) \Big/ \left(\sigma_L^2 W_6\right) = (X_6 + 1)/W_6 \,,$$

which yields the conclusion.

9.3 It is easy to deduce from (9.51) that $V_T^{(A)}(\theta) = U_T/V_T$, where

$$V_T = \frac{1}{T}\sum_{j=1}^{T}(y_j - y_{j-1})^2 \longrightarrow \sigma_S^2 \qquad \text{in probability}\,,$$

$$\begin{aligned} \mathcal{L}(U_T) \quad &= \quad \mathcal{L}\left(-\frac{\theta^2}{T^2}\sum_{j=2}^{T} y_{j-1}^2 + \frac{\theta}{T}\sum_{j=2}^{T}(y_j - y_{j-1})^2 - \frac{\theta}{T}y_T^2\right) \\ &\longrightarrow \quad \mathcal{L}\left(-\theta^2\sigma_L^2\int_0^1 Y^2(t)dt - \theta\sigma_L^2 Y^2(1) + \theta\sigma_S^2\right) \,. \end{aligned}$$

Thus we can establish (9.81) for Model A.

10.1 The LBI test rejects H_0 when

$$S_T = -\frac{\left(y - X\hat{\beta}\right)' \left.\dfrac{d\Omega^{-1}(\rho)}{d\rho}\right|_{\rho=0} \left(y - X\hat{\beta}\right)}{\hat{\sigma}^2} > c\,,$$

where $\hat{\beta} = (X'X)^{-1}X'y\,, \hat{\sigma}^2 = (y - X\hat{\beta})'(y - X\hat{\beta})/T$ and $\Omega(\rho) = I_T + \rho CC'$. It is easily seen that the above test is equivalent to the test based on U_T.

10.2 Since $y'My = (\varepsilon + C\xi)'M(\varepsilon + C\xi)$ and $\varepsilon + C\xi \sim \mathrm{N}(0,\, \sigma_\varepsilon^2(I_T + \rho CC'))$, we have

$$\begin{aligned} \mathcal{L}\left(\frac{1}{T}y'My\right) \quad &= \quad \mathcal{L}\left(\frac{\sigma_\varepsilon^2}{T}Z'(I_T + \rho CC')^{\frac{1}{2}}M(I_T + \rho CC')^{\frac{1}{2}}Z\right) \\ &= \quad \mathcal{L}\left(\frac{\sigma_\varepsilon^2}{T}Z'MZ + \frac{c^2\sigma_\varepsilon^2}{T^3}Z'MCC'MZ\right)\,, \end{aligned}$$

where $Z \sim \mathrm{N}(0,\, I_T)$. It holds that

$$\frac{1}{T}Z'MZ \longrightarrow 1 \qquad \text{in probability,} \qquad \frac{1}{T^3}Z'MCC'MZ = O_p\left(\frac{1}{T}\right)\,.$$

Thus (9.88) is established.

10.3 Let us consider

$$B_T = \frac{1}{T}C'MC = \frac{1}{T}\left[C'C - C'X(X'X)^{-1}X'C\right],$$

where $X = (e, d)$. It is seen that $K(s,t)$ in (9.91) satisfies $\lim_{T\to\infty} \max_{j,k} |\, B_T(j,k) - K(j/T, k/T)\,| = 0$. Moreover the symmetric and continuous kernel $K(s,t)$ is shown to be positive definite. Thus (9.90) follows from Theorem 5.13.

10.4 Because of Theorem 5.13 we have only to prove that the FD of $K(s,t)$ is given by (9.94). The integral equation (5.10) is shown to be equivalent to

$$\begin{aligned} f(t) &= c_1 \cos\sqrt{\lambda}t + c_2 \sin\sqrt{\lambda}t + 6a\,, \\ f(0) &= f(1) = 0\,, \qquad a = \int_0^1 (s - s^2) f(s)ds\,. \end{aligned}$$

Then the approach taken in Section 4 of Chapter 5 leads us to obtain the FD of $K(s,t)$ as in (9.94).

10.5 The LBI test, if it exists, rejects H_0 when

$$R_T = -\frac{y'M(C')^{-1}\left.\dfrac{d\Sigma^{-1}(\alpha)}{d\alpha}\right|_{\alpha=1} C^{-1}My}{y'My} < c\,,$$

where $\Sigma^{-1}(\alpha) = C'(\alpha)C(\alpha)$. Since

$$\begin{aligned} \left.\frac{d\Sigma^{-1}(\alpha)}{d\alpha}\right|_{\alpha=1} &= -\Sigma^{-1}(1)\left.\frac{d\Sigma(\alpha)}{d\alpha}\right|_{\alpha=1}\Sigma^{-1}(1) \\ &= -C'(I_T - ee')C\,, \end{aligned}$$

it is seen that $R_T = y'M(I_T - ee')My/y'My = 1$. Thus we consider the LBIU test which rejects H_0 when

$$S_T = -\frac{y'M(C')^{-1}\left.\dfrac{d^2\Sigma^{-1}(\alpha)}{d\alpha^2}\right|_{\alpha=1} C^{-1}My}{y'My} > c\,.$$

Since it can be shown that

$$\left.\frac{d^2\Sigma^{-1}(\alpha)}{d\alpha^2}\right|_{\alpha=1} = 2C'(I_T - ee')^2C - 2C'(CC' - ee')C\,,$$

it is seen that the above test is equivalent to the test based on V_T.

10.6 Note first that $\mathcal{L}(y'My) = \mathcal{L}\left(\varepsilon' MC\left(C'(\alpha)C(\alpha)\right)^{-1} C'M\varepsilon\right)$, where

$$
\begin{aligned}
\text{(S16)} \qquad \left(C'(\alpha)C(\alpha)\right)^{-1} &= \alpha\left(C'C\right)^{-1} + (1-\alpha)^2 I_T + \alpha(1-\alpha)e_1e_1' \\
&= \left(1-\frac{c}{T}\right)\left(C'C\right)^{-1} + \frac{c^2}{T^2}I_T + \frac{c}{T}\left(1-\frac{c}{T}\right)e_1e_1' \,.
\end{aligned}
$$

Then we obtain

$$
\mathcal{L}\left(\frac{1}{T}y'My\right) = \mathcal{L}\left(\frac{1}{T}\varepsilon' M\varepsilon + o_p(1)\right) \,,
$$

which establishes (9.99).

10.7 Using (S16) we can deduce that

$$
\mathcal{L}\left(\frac{1}{T^2\sigma_\varepsilon^2}\, y'MCC'My\right) = \mathcal{L}\left(\frac{1}{T^2}Z'C'M\left(I_T + \frac{c^2}{T^2}CC'\right)MCZ + o_p(1)\right) \,.
$$

Thus we can establish the weak convergence result (9.100) from (9.89) and (9.90).

Chapter 10.

2.1 The LM principle yields the LBI test which rejects H_0 when

$$
-\frac{y'\bar{M}'\dfrac{d}{d\alpha}C'(\alpha)C(\alpha)\Big|_{\alpha=1}\bar{M}y}{y'\bar{M}'C'C\bar{M}y} < c \,,
$$

where $\bar{M} = I_T - X(X'C'CX)^{-1}X'C'C$. Since $dC'(\alpha)C(\alpha)/\,d\alpha|_{\alpha=1} = C'ee'C - C'C$ and $C\bar{M} = \tilde{M}C$ with $\tilde{M}^2 = \tilde{M}$, the above test implies (10.6).

2.2 Noting that $\tilde{M}Cy = \tilde{M}CC^{-1}(\alpha_0)\varepsilon$ and $CC^{-1}(\alpha_0) = I_T + (1-\alpha_0)(C - I_T)$ we have

$$
\begin{aligned}
\frac{1}{T}y'C'\tilde{M}Cy &= \frac{1}{T}\varepsilon'\left[I_T + \frac{c}{T}(C' - I_T)\right]\tilde{M}\left[I_T + \frac{c}{T}(C - I_T)\right]\varepsilon \\
&= \frac{1}{T}\varepsilon'\left[\tilde{M} + \frac{c}{T}\left\{\tilde{M}(C - I_T) + (C' - I_T)\tilde{M}\right\}\right. \\
&\qquad\qquad \left. + \frac{c^2}{T^2}(C' - I_T)\tilde{M}(C - I_T)\right]\varepsilon \,.
\end{aligned}
$$

Here it holds that $\operatorname{plim}\left(\varepsilon'\tilde{M}\varepsilon\Big/T\right)=\sigma^2$, while the other terms converge in probability to 0. Thus we establish (10.7).

2.3 Let us consider

$$\begin{aligned}\mathcal{L}\left(\frac{1}{T\sigma^2}y'C'\tilde{M}ee'\tilde{M}Cy\right) &= \mathcal{L}\left(\frac{1}{T\sigma^2}\left(e'\tilde{M}CC^{-1}(\alpha_0)\varepsilon\right)^2\right)\\ &= \mathcal{L}\left(\left(\frac{1}{\sqrt{T}}\sum_{j=1}^{T}a_jZ_j\right)^2\right)\to A\chi^2(1)\,,\end{aligned}$$

where $\{Z_j\}\sim\mathrm{NID}(0,1)$ and

$$\begin{aligned}A &= \lim_{T\to\infty}\frac{1}{T}\sum_{j=1}^{T}a_j^2=\lim_{T\to\infty}\frac{1}{T}e'\tilde{M}C\left(C'(\alpha_0)C(\alpha_0)\right)^{-1}C'\tilde{M}e\\ &= \lim_{T\to\infty}\frac{1}{T}e'\tilde{M}C\left[\left(C'C\right)^{-1}+\frac{c}{T}e_1e_1'+\frac{c^2}{T^2}I_T\right]C'\tilde{M}e\end{aligned}$$

The computation of the value A for each model establishes Theorem 10.1.

2.4 It follows from (8.42) that the limiting c.f. of $X_T=V_{T1}(\theta)+\theta$ for $c=0$ is given by

$$\phi(u)=\left[\left(\cos\theta\sqrt{2iu-1}-\sqrt{2iu-1}\sin\theta\sqrt{2iu-1}\right)\Big/e^{\theta}\right]^{-\frac{1}{2}}\,.$$

Thus $P\left(V_{T1}(\theta)\le x\right)=P\left(X_T/\theta^2\le(\theta+x)/\theta^2\right)$ yields (10.11).

2.5 When $\theta=c$, the kernel $K(s,t;\theta)$ in (10.10) takes the form :

$$K(s,t)=c+c^2-c^2\max(s,t)\,,$$

whose FD is found to be

$$D(\lambda)=\cos c\sqrt{\lambda}-\sqrt{\lambda}\sin c\sqrt{\lambda}\,.$$

Thus the limiting c.f. of $Y_T=V_{T1}(c)+c$ is given by $(D(2iu))^{-\frac{1}{2}}$. Since $P\left(V_{T1}(c)\ge x\right)=P\left(Y_T/c^2\ge(c+x)/c^2\right)$, (10.12) is established.

2.6 Let $P_{lm}(j\,|\,k)$ be the (l,m)-th element of $P(j\,|\,k)$. Then the Kalman filter algorithm yields

$$P(j\,|\,j-1) = \begin{pmatrix} P_{22}(j-1\,|\,j-1)+\sigma^2 & \sigma^2 \\ \sigma^2 & \sigma^2 \end{pmatrix},$$

$$P(j\,|\,j) = \begin{pmatrix} 0 & 0 \\ 0 & P_{22}(j\,|\,j-1) - \dfrac{P_{12}^2(j\,|\,j-1)}{P_{11}(j\,|\,j-1)} \end{pmatrix}.$$

Thus we obtain

$$P_{11}(j\,|\,j-1) = 2\sigma^2 - \frac{\sigma^4}{P_{11}(j-1\,|\,j-2)},$$

where $P_{11}(1\,|\,0) = V(y_1) = 2\sigma^2$. We now have $P_{11}(j\,|\,j-1) = (j+1)\sigma^2/j$. Putting $\beta(j\,|\,k) = (\beta_1(j\,|\,k), \beta_2(j\,|\,k))'$ we can also derive

$$\beta(j\,|\,j-1) = \begin{pmatrix} -\beta_2(j-1\,|\,j-1) \\ 0 \end{pmatrix},$$

$$\begin{aligned} \beta(j\,|\,j) &= \beta(j\,|\,j-1) + \begin{pmatrix} 1 \\ P_{21}(j\,|\,j-1)/P_{11}(j\,|\,j-1) \end{pmatrix} (y_j - \beta_1(j\,|\,j-1)) \\ &= \begin{pmatrix} y_j \\ \dfrac{j}{j+1}\,(y_j + \beta_2(j-1\,|\,j-1)) \end{pmatrix}, \end{aligned}$$

so that

$$\begin{aligned} \beta_2(j\,|\,j) &= \frac{j}{j+1}\beta_2(j-1\,|\,j-1) + \frac{j}{j+1}\,y_j \\ &= \frac{1}{j+1}(y_1 + 2y_2 + \cdots + jy_j), \end{aligned}$$

$$\begin{aligned} y_j - a'\beta(j\,|\,j-1) &= y_j + \beta_2(j-1\,|\,j-1) \\ &= \frac{1}{j}(y_1 + 2y_2 + \cdots + jy_j). \end{aligned}$$

Therefore we obtain

$$
\begin{aligned}
y'\Omega^{-1}y &= \sigma^2 \sum_{j=1}^{T} \frac{(y_j - a'\beta(j\,|\,j-1))^2}{a'P(j\,|\,j-1)a} \\
&= \sum_{j=1}^{T} \frac{1}{j(j+1)}(y_1 + 2y_2 + \cdots + jy_j)^2 .
\end{aligned}
$$

2.7 Noting that $B(\alpha)B'(\alpha) = \Omega(\alpha) = \alpha\Omega + (1-\alpha)^2 I_T$ we have

$$
\begin{aligned}
\mathcal{L}\left(\frac{1}{T}\tilde{\eta}'\Omega^{-1}\tilde{\eta}\right) &= \mathcal{L}\left(\frac{1}{T}\varepsilon' M\Omega^{-\frac{1}{2}}\left(\alpha\Omega + (1-\alpha)^2 I_T\right)\Omega^{-\frac{1}{2}}M\varepsilon\right) \\
&= \mathcal{L}\left(\frac{\alpha}{T}\varepsilon' M\varepsilon + \frac{c^2}{T^3}\varepsilon' M\Omega^{-1}M\varepsilon\right) ,
\end{aligned}
$$

where $M = I_T - \Omega^{-\frac{1}{2}}X(X'\Omega^{-1}X)^{-1}X'\Omega^{-\frac{1}{2}} = M^2$. It clearly holds that $\operatorname{plim}(\alpha\varepsilon' M\varepsilon/T) = \sigma^2$, while $\varepsilon' M\Omega^{-1}M\varepsilon = O_p(T^2)$, which establishes (10.21).

2.8 The case of Model A can be easily proved. For Model B we obtain

$$
A_T = \Omega^{-1} - \Omega^{-1}ee'\Omega^{-1}\Big/ e'\Omega^{-1}e ,
$$

where

$$
\begin{aligned}
\Omega^{-1}e &= \left(C'C - \frac{1}{T+1}C'ee'C\right)e = C'd - \frac{T}{2}C'e , \\
e'\Omega^{-1}e &= e'C'Ce - \frac{1}{T+1}\left(e'Ce\right)^2 = \frac{T(T+1)(T+2)}{12} .
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
A_T(j,k) = {} & \min(j,k) - \frac{jk}{T+1} \\
& - \frac{3\left\{(T+j)(T-j+1) - T(T-j+1)\right\}\left\{(T+k)(T-k+1) - T(T-k+1)\right\}}{T(T+1)(T+2)} .
\end{aligned}
$$

Then it can be checked that $A_T(j,k)/T$ has the uniform limit $K_B(s,t)$ in the sense of (10.23). The integral equation (5.10) with $K = K_B$ is shown to be equivalent to

$$
\begin{aligned}
f(t) &= c_1 \cos\sqrt{\lambda}\,t + c_2 \sin\sqrt{\lambda}\,t + 6a , \\
f(0) &= f(1) = 0 , \qquad a = \int_0^1 (t - t^2)f(t)dt .
\end{aligned}
$$

Then the Fredholm approach yields the FD $D_B(\lambda)$ of K_B. The case of Model C can be similarly proved. As for Model D it is not hard to find the kernel $K_D(s,t)$. To obtain the FD of K_D we need to evaluate the determinant of a 4×4 matrix, which we have done by REDUCE, and arrive at $D_D(\lambda)$ in the theorem.

2.9 It follows from (8.15) and (8.18) that the limiting c.f. of $X_T = V_{T2}(\theta) + \theta$ for $c = 0$ is given by

$$\phi(u) = \left[\frac{\sin \theta\sqrt{2iu-1}}{\theta\sqrt{2iu-1}} \Big/ \frac{\sinh \theta}{\theta} \right]^{-\frac{1}{2}} .$$

Thus $P(V_{T2}(\theta) \le x) = P(X_T/\theta^2 \le (\theta + x)/\theta^2)$ yields (10.28).

2.10 When $\theta = c$, it follows from (10.27) that

$$\mathcal{L}\left(V_{T2}(c)\right) \longrightarrow \mathcal{L}\left(c^2 \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} Z_n^2 - c\right) ,$$

which evidently yields (10.29).

4.1 It follows from (10.26) that, under $\alpha = 1 - (\theta/T)$ and $\alpha_0 = 1$,

$$\begin{aligned} P\left(V_{T2}(\theta) \ge x\right) &= P\left(\frac{x}{T} y'\Omega^{-1}y - y'\left(\Omega^{-1} - \Omega^{-1}(\alpha)\right)y \le 0\right) \\ &= P\left(\sum_{j=1}^{T} \left(\left(\frac{x}{T} - 1\right) + \frac{2 - 2\delta_j}{1 + \alpha^2 - 2\alpha\delta_j}\right) Z_j^2 \le 0\right) , \end{aligned}$$

where $\delta_j = \cos(j\pi/(T+1))$. Then the upper 5% point x can be computed for each α and T by Imhof's formula described in (7.92).

4.2 Using (10.28) we first obtain the upper 5% point x for each $\theta = c$. Then the limiting power envelope can be computed following (10.29) for each combination of c and x.

4.3 Using (10.28) we first obtain the upper 5% point x for a fixed θ at which the point optimal test is conducted. It follows from (10.27) and (8.18) that the limiting c.f. of $X_T = V_{2T}(\theta) + \theta$ as $T \to \infty$ under $\alpha = 1 - (\theta/T)$ and $\alpha_0 = 1 - (c/T)$ is given by

$$\phi(u) = \left[\frac{\sin\sqrt{a+b}}{\sqrt{a+b}} \frac{\sin\sqrt{a-b}}{\sqrt{a-b}} \Big/ \frac{\sinh\theta}{\theta} \right]^{-\frac{1}{2}} ,$$

where

$$a = \frac{\theta^2(2iu-1)}{2}, \qquad b = \frac{\theta\sqrt{\theta^2(2iu-1)^2 + 8ic^2u}}{2}.$$

Thus the limiting powers can be computed as

$$\begin{aligned} \lim_{T\to\infty} P\left(V_{T2}(\theta) \ge x\right) &= \lim_{T\to\infty} P\left(\frac{X_T}{\theta^2} \ge \frac{\theta + x}{\theta^2}\right) \\ &= 1 - \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{1 - \exp\left\{-\dfrac{iu(\theta+x)}{\theta^2}\right\}}{iu}\,\phi\left(\frac{u}{\theta^2}\right)\right] du\,. \end{aligned}$$

5.1 Since $y \sim \mathrm{N}\left((\bar{X} \otimes I_m)\beta,\ \sigma^2 \left(C'(\alpha_m)C(\alpha_m)\right)^{-1} \otimes I_m\right)$, the LBI test rejects H_0 when

$$-\frac{y'\left[\left(\bar{M}'\dfrac{d}{d\alpha}C'(\alpha)C(\alpha)\Big|_{\alpha=1}\bar{M}\right) \otimes I_m\right] y}{y'\left[\left(\bar{M}'C'C\bar{M}\right) \otimes I_m\right] y} < c\,,$$

where $\bar{M} = I_N - \bar{X}\left(\bar{X}'C'C\bar{X}\right)^{-1}\bar{X}'C'C$. Substituting $dC'(\alpha)C(\alpha)/\,d\alpha\big|_{\alpha=1} = C'ee'C - C'C$ we obtain the LBI statistic S_{N1}.

5.2 Since the limiting distribution of $X_N = V_{N1}(\theta) + m\theta$ is the m-fold convolution of that of $V_{T1}(\theta) + \theta$ in (10.11), (10.47) follows immediately from (10.11) by noting that $P(V_{N1} \le x) = P(X_N/\theta^2 \le (m\theta + x)/\theta^2)$.

5.3 The limiting distribution of $Y_N = V_{N1}(c) + cm$ is the m-fold convolution of that of $V_{T1}(c) + c$ in (10.12). Thus (10.48) follows from (10.12) by noting that $P\left(V_{N1}(c) \ge x\right) = P\left(Y_N/\,c^2 \ge (cm + x)/\,c^2\right)$.

5.4 Since $y \sim \mathrm{N}\left(\left(\bar{X} \otimes I_m\right)\beta,\ \sigma^2\Omega(\alpha_m) \otimes I_m\right)$, the LBIU test rejects H_0 when

$$-\frac{y'\left[\left(\tilde{N}'\dfrac{d^2}{d\alpha^2}\Omega^{-1}(\alpha)\bigg|_{\alpha=1}\tilde{N}\right) \otimes I_m\right]y}{y'\left[\left(\tilde{N}'\Omega^{-1}\tilde{N}\right) \otimes I_m\right]y} > c\,.$$

Since $d^2\Omega^{-1}(\alpha)\Big/ d\alpha^2\Big|_{\alpha=1} = 2\Omega^{-1} - 2\Omega^{-2}$, S_{N2} is shown to be the LBIU statistic.

5.5 The limiting distribution of $X_N = V_{N2}(\theta) + m\theta$ is the m-fold convolution of that of $V_{T2}(\theta) + \theta$ in (10.28). Thus (10.54) follows from (10.28).

5.6 The limiting distribution of $Y_N = V_{N2}(c) + cm$ is the m-fold convolution of that of $V_{T2}(c) + c$ in (10.29). Thus (10.55) follows from (10.29).

7.1 Since it holds that

$$\begin{aligned}\mathcal{L}\left(\frac{y'My}{T\sigma_\varepsilon^2}\right) &= \mathcal{L}\left(\frac{1}{T}Z'M(I_T + \rho DCC'D)MZ\right) \\ &= \mathcal{L}\left(\frac{1}{T}Z'MZ + \frac{c^2}{T^{2m+3}}Z'MDCC'DMZ\right),\end{aligned}$$

where $Z \sim \mathrm{N}(0, I_T)$, we establish (10.69) noting that $\mathrm{plim}(Z'MZ/T) = 1$ and $Z'MDC$ $C'DMZ = O_p(T^{2m+2})$.

7.2 It is easy to deduce that the (j,k)-th element of $C'DMDC$ is given by

$$B_T(j,k) = \sum_{l=\max(j,k)}^{T} l^{2m} - \sum_{l=j}^{T} l^{2m} \sum_{l=k}^{T} l^{2m} \Bigg/ \sum_{l=1}^{T} l^{2m}\,.$$

Thus $B_T(j,k)/T^{2m+1}$ converges uniformly to $K(s,t;m)$, which proves (10.70) because of Theorem 5.13. The associated FD was earlier obtained in (5.64).

7.3 The (j,k)-th element of $C'MC$ is given by

$$\begin{aligned}B_T(j,k) &= T + 1 - \max(j,k) - \frac{1}{T\sum_{l=1}^{T} l^{2m} - \left(\sum_{l=1}^{T} l^m\right)^2} \\ &\quad \times \Bigg[(T-k+1)\{(T-j+1)\sum_{l=1}^{T} l^{2m} - \sum_{l=j}^{T} l^m \sum_{l=1}^{T} l^m\}\end{aligned}$$

$$+\sum_{l=k}^{T} l^m \{T \sum_{l=j}^{T} l^m - (T-j+1)\sum_{l=1}^{T} l^m\}\Bigg] \,.$$

Then it can be checked that $B_T(j,k)/T$ converges uniformly to $K(s,t;m)$. The associated FD's for $m=1$ and 2 are available in Theorem 10.2. Consider the case of $m=3$. The integral equation (5.10) with $K(s,t)$ replaced by $K(s,t;3)$ is shown to be equivalent to

$$\begin{aligned} f(t) &= c_1 \cos\sqrt{\lambda}\,t + c_2 \sin\sqrt{\lambda}\,t + at^2 - 2a/\lambda\,, \\ f(0) &= f(1) = 0, \qquad a = \frac{28}{3}\int_0^1 t(1-t^3)f(t)dt\,. \end{aligned}$$

Then we obtain $D(\lambda;3)$ after some algebra. The case of $m=4$ can be dealt with similarly.

7.4 Since it can be shown that $\text{plim}(y'My/T) = \sigma_\varepsilon^2$ as $T \to \infty$ under $\rho = c^2/T^2$, we concentrate on

$$\mathcal{L}\left(\frac{1}{T^2\sigma_\varepsilon^2} y'MCC'My\right) = \mathcal{L}\left(\frac{1}{T^2}Z'\left(C'MC + \frac{c^2}{T^2}\left(C'MC\right)^2\right)Z\right)\,,$$

where $Z \sim \mathrm{N}(0, I_T)$. Evaluating the (j,k)-th element of

$$C'MC = C'C - C'(e,d,f)\left((e,d,f)'(e,d,f)\right)^{-1}(e,d,f)'C\,,$$

we obtain the kernel $K(s,t)$. Then the Fredholm approach yields the FD given in the theorem.

7.5 It follows from (10.80) that the limiting c.f. of $V_T(\theta)/\theta^2$ for $c=0$ is given by

$$\begin{aligned} \phi(u) &= \prod_{n=1}^{\infty}\left[1 - \frac{2iu}{\left(n-\frac{1}{2}\right)^2\pi^2 + \theta^2}\right]^{-\frac{1}{2}} \\ &= \prod_{n=1}^{\infty}\left[\left\{1 - \frac{2iu-\theta^2}{\left(n-\frac{1}{2}\right)^2\pi^2}\right\}\Bigg/\left\{1 + \frac{\theta^2}{\left(n-\frac{1}{2}\right)^2\pi^2}\right\}\right]^{-\frac{1}{2}} \\ &= \left[\cos\sqrt{2iu-\theta^2}\Big/\cosh\theta\right]^{-\frac{1}{2}}\,. \end{aligned}$$

Thus (10.81) follows from $P\left(V_T(\theta) \le x\right) = P\left(V_T(\theta)/\theta^2 \le x/\theta^2\right)$.

7.6 When $\theta = c$, it follows from (10.80) that

$$\mathcal{L}\left(V_T(c)\right) \longrightarrow \mathcal{L}\left(c^2 \sum_{n=1}^{\infty} \frac{1}{\left(n-\frac{1}{2}\right)^2 \pi^2} Z_n^2\right).$$

Then $P\left(V_T(c) \ge x\right) = P\left(V_T(c)/\,c^2 \ge x/\,c^2\right)$ implies (10.82).

7.7 It follows from (10.80) that the limiting c.f. of $V_T(\theta)$ is given by

$$\begin{aligned}
\phi(u) &= \prod_{n=1}^{\infty} \left[1 - \frac{2iu\theta^2\left(\left(n-\frac{1}{2}\right)^2 \pi^2 + c^2\right)}{\left(n-\frac{1}{2}\right)^2 \pi^2 \left(\left(n-\frac{1}{2}\right)^2 \pi^2 + \theta^2\right)}\right]^{-\frac{1}{2}} \\
&= \prod_{n=1}^{\infty} \left[\frac{1 + \dfrac{(1-2iu)\,\theta^2}{\left(n-\frac{1}{2}\right)^2 \pi^2} - \dfrac{2ic^2u\theta^2}{\left(n-\frac{1}{2}\right)^4 \pi^4}}{1 + \dfrac{\theta^2}{\left(n-\frac{1}{2}\right)^2 \pi^2}}\right]^{-\frac{1}{2}} \\
&= \left(\cos\sqrt{a+b}\,\cos\sqrt{a-b}\Big/\cosh\theta\right)^{-\frac{1}{2}},
\end{aligned}$$

where

$$a = \frac{\theta^2}{2}\left(2iu-1\right), \qquad b = \frac{\theta}{2}\sqrt{\theta^2\left(2iu-1\right)^2 + 8ic^2u}\,.$$

Then the limiting powers of the POI test conducted at θ under the $100\gamma\%$ significance level can be computed as

$$\lim_{T\to\infty} \mathrm{P}\left(\frac{V_T(\theta)}{\theta^2} \ge \frac{x}{\theta^2}\right) = 1 - \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\left[\frac{1-\exp\left(-\frac{iux}{\theta^2}\right)}{iu}\phi\left(\frac{u}{\theta^2}\right)\right] du\,,$$

where x is the upper $100\gamma\%$ point of the limiting distribution in (10.81).

Chapter 11.

2.1 Let us construct the partial sum process :

$$\text{(S17)} \qquad Y_T(t) = \frac{1}{\sqrt{T}}\, y_j + T\left(t - \frac{j}{T}\right)\frac{1}{\sqrt{T}}\, u_j\,, \qquad \left(\frac{j-1}{T} \le t \le \frac{j}{T}\right).$$

Then it follows that $\mathcal{L}\left(Y_T\right) \to \mathcal{L}\left(Aw\right)$ and

$$\begin{aligned} \mathcal{L}\left(\frac{1}{T^2}\sum_{j=1}^{T} y_j y_j'\right) &= \mathcal{L}\left(\frac{1}{T}\sum_{j=1}^{T} Y_T\left(\frac{j}{T}\right) Y_T'\left(\frac{j}{T}\right)\right) \\ &\longrightarrow \mathcal{L}\left(A\int_0^1 w(t)w'(t)dtA'\right). \end{aligned}$$

The continuous mapping theorem now establishes (11.12) for $k=1$. If we construct, for $(j-1)/T \le t \le j/T$,

$$\text{(S18)} \qquad \tilde{Y}_T(t) = \frac{1}{\sqrt{T}}\left(y_j - \bar{y}\right) + T\left(t - \frac{j}{T}\right)\frac{1}{\sqrt{T}}\, u_j\,,$$

we have that $\mathcal{L}\left(\tilde{Y}_T\right) \to \mathcal{L}\left(A\tilde{w}\right)$ and

$$\begin{aligned} \mathcal{L}\left(\frac{1}{T^2}\sum_{j=1}^{T}\left(y_j-\bar{y}\right)\left(y_j-\bar{y}\right)'\right) &= \mathcal{L}\left(\frac{1}{T}\sum_{j=1}^{T} \tilde{Y}_T\left(\frac{j}{T}\right) \tilde{Y}_T'\left(\frac{j}{T}\right)\right) \\ &\longrightarrow \mathcal{L}\left(A\int_0^1 \tilde{w}(t)\tilde{w}'(t)dtA'\right), \end{aligned}$$

where $\tilde{w}(t) = w(t) - \int_0^1 w(s)ds$. Then (11.12) also holds for $k=2$ because of the continuous mapping theorem.

2.2 For $k=1$ it follows from Theorem 4.4 that

$$\begin{aligned} E\left(e^{\theta X_k}\right) &= E\left[\exp\left\{\theta\int_0^1 w'(t)Hw(t)dt\right\}\right] \\ &= \prod_{a=1}^{2} E\left[\exp\left\{\theta\delta_a\int_0^1 w_a^2(t)dt\right\}\right], \end{aligned}$$

where $w(t) = (w_1(t), w_2(t))'$ is the two-dimensional standard Brownian motion. Then we obtain (11.17) from (4.10). The case of $k=2$ can be proved similarly.

2.3 Let $\nu_k(n)$ be the n-th order raw moment of $F_k(x)$ in (11.15). Then, we have, from (1.39),

$$\text{(S19)} \qquad \nu_k(n) = \frac{1}{(n-1)!}\int_0^\infty \theta_2^{n-1}\left.\frac{\partial^n \psi_k\left(\theta_1, -\theta_2\right)}{\partial\theta_1^n}\right|_{\theta_1=0} d\theta_2\,,$$

where

$$\begin{aligned}
\psi_1\left(\theta_1\,,-\theta_2\right) &= \left[\cos\sqrt{a+b}\ \cos\sqrt{a-b}\right]^{-\frac{1}{2}}\,, \\
\psi_2\left(\theta_1\,,-\theta_2\right) &= \left[\frac{\sin\sqrt{a+b}}{\sqrt{a+b}}\ \frac{\sin\sqrt{a-b}}{\sqrt{a-b}}\right]^{-\frac{1}{2}}\,, \\
a = -\theta_2 A_1' A_1 + \theta_1 A_1' A_2\,, &\qquad b = \sqrt{a^2 + \theta_1^2 \mid A\mid^2}\,.
\end{aligned}$$

Using any computerized algebra we can easily obtain partial derivatives of ψ_k. Then we compute, for instance,

$$\nu_1(1) = \frac{1}{2} A_1' A_2 \int_0^\infty \frac{\sinh\sqrt{2\theta A_1' A_1}}{\sqrt{2\theta A_1' A_1}} \left(\cosh\sqrt{2\theta A_1' A_1}\right)^{-\frac{3}{2}} d\theta = \frac{A_1' A_2}{A_1' A_1}\,.$$

Finally we can obtain $\mu_k(n)$ from $\nu_k(n)$.

2.4 Mercer's theorem (Theorem 5.2) gives us

$$\text{(S20)} \qquad F_k(x) = P\left(\sum_{n=1}^\infty \frac{1}{\lambda_{kn}}\left(\delta_1(x) X_n^2 + \delta_2(x) Y_n^2\right) \ge 0\right)\,,$$

where $(X_n\,, Y_n)' \sim \mathrm{NID}(0, I_2)\,,\ \lambda_{1n} = \left(n - \frac{1}{2}\right)^2 \pi^2$ and $\lambda_{2n} = n^2\pi^2$, while $\delta_1(x)$ and $\delta_2(x)$ are the eigenvalues of the matrix H given in (11.19). Then it can be checked easily that $F_k(x+\mu) = 1 - F_k(-x+\mu)$.

2.5 It follows from (S20) that

$$\begin{aligned}
G_k(x) &= F_k\left(\sigma_k x + \mu\right) \\
&= P\left(\sum_{n=1}^\infty \frac{1}{\lambda_{kn}}\left(\delta_1\left(\sigma_k x + \mu\right) X_n^2 + \delta_2\left(\sigma_k x + \mu\right) Y_n^2\right) \ge 0\right)\,,
\end{aligned}$$

where $\sigma_k = \sqrt{\mu_k(2)}$. Since

$$\begin{aligned}
\delta_1\left(\sigma_k x + \mu\right),\, \delta_2\left(\sigma_k x + \mu\right) &= \frac{1}{2}\left[\sigma_k x A_1' A_1 \pm \sqrt{\left(\sigma_k x A_1' A_1\right)^2 + \mid A\mid^2}\right] \\
&= \frac{\mid\det(A)\mid}{2}\left(\sqrt{a_k} x \pm \sqrt{a_k x^2 + 1}\right)\,,
\end{aligned}$$

where a_k is defined in Corollary 11.1, (11.22) is seen to hold.

2.6 We first note that $\hat{v}_j = y_{2j} - \hat{\beta}_1 y_{1j}$; hence

$$\frac{1}{T^2}\sum_{j=1}^{T}\hat{v}_j^2 = \frac{1}{T^2}\sum_{j=1}^{T}y_{2j}^2 - \hat{\beta}_1^2\frac{1}{T^2}\sum_{j=1}^{T}y_{1j}^2 \,.$$

Since it holds that

$$\mathcal{L}\left(\hat{\beta}_1\,,\ \frac{1}{T^2}\sum_{j=1}^{T}y_{1j}^2\,,\ \frac{1}{T^2}\sum_{j=1}^{T}y_{2j}^2\,,\ \frac{1}{T^2}\sum_{j=1}^{T}\hat{v}_j^2\right) \longrightarrow \mathcal{L}\left(\zeta\,, A_1'W_1A_1\,,\ A_2'W_1A_2\,,\ A_2'W_1A_2 - \zeta^2 A_1'W_1A_1\right)\,,$$

this proves (11.23) and (11.24). Moreover

$$\frac{1}{T}\sum_{j=2}^{T}(\hat{v}_j - \hat{v}_{j-1})^2 = \left(\hat{\beta}_1\,, -1\right)\frac{1}{T}\sum_{j=1}^{T}u_j u_j' \begin{pmatrix} \hat{\beta}_1 \\ -1 \end{pmatrix}\,,$$

which yields (11.25).

3.1 Since Y_1' can be expressed as $Y_1' = \Xi_1' C'$, we have

$$\begin{aligned} \text{vec}\,(Y_1') &= \text{vec}\,(\Xi_1' C') \\ &= (C \otimes I_q)\ \text{vec}\,(\Xi_1')\,. \end{aligned}$$

Noting that vec $(\Xi_1') \sim \mathrm{N}\,(0\,, I_T \otimes \Sigma_{11})$ we obtain (11.33). Moreover it can be shown that

$$\begin{pmatrix} \text{vec}\,(Y_1') \\ \\ \Xi_2 \end{pmatrix} \sim \mathrm{N}\left(0\,, \begin{pmatrix} (CC') \otimes \Sigma_{11} & C \otimes \Sigma_{12} \\ \\ C' \otimes \Sigma_{21} & I_T \otimes \Sigma_{22} \end{pmatrix}\right)\,,$$

which proves (11.34)

3.2 It is easy to deduce that

$$\begin{aligned} \frac{1}{T^2}Y_1'P_{-1}Y_1 &= \frac{1}{T^2}Y_1'Y_{-1}\left(Y_{-1}'Y_{-1}\right)^{-1}Y_{-1}'Y_1 \\ &= \frac{1}{T^2}Y_1'(Y_1 - \Xi_1)\left\{\frac{1}{T^2}(Y_1 - \Xi_1)'(Y_1 - \Xi_1)\right\}^{-1}\frac{1}{T^2}(Y_1 - \Xi_1)'Y_1 \\ &= \frac{1}{T^2}Y_1'Y_1 + o_p(1)\,, \\ \frac{1}{T^2}Y_1'M_1Y_1 &= \frac{1}{T^2}Y_1'Y_1 - \frac{1}{T}Y_1'\Xi_1\left(\frac{1}{T}\Xi_1'\Xi_1\right)^{-1}\frac{1}{T}\Xi_1'Y_1 \times \frac{1}{T} \\ &= \frac{1}{T^2}Y_1'Y_1 + o_p(1)\,. \end{aligned}$$

3.3 We can show easily that

$$\begin{aligned}
\mathcal{L}\left(\frac{1}{T}Y_1'P_{-1}\Xi_2\right) &= \mathcal{L}\left(\frac{1}{T}Y_1'Y_{-1}\left(Y_{-1}'Y_{-1}\right)^{-1}Y_{-1}'\Xi_2\right)\\
&= \mathcal{L}\left(\frac{1}{T}Y_{-1}'\Xi_2 + o_p(1)\right)\\
&\longrightarrow \mathcal{L}\left(U_1+U_2\right),\\
\mathcal{L}\left(\frac{1}{T}Y_1'M_1\Xi_2\right) &= \mathcal{L}\left(\frac{1}{T}Y_1'\Xi_2 - \frac{1}{T}Y_1'\Xi_1\left(\frac{1}{T}\Xi_1'\Xi_1\right)^{-1}\frac{1}{T}\Xi_1'\Xi_2\right)\\
&\longrightarrow \mathcal{L}\left(U_1+U_2+\Sigma_{12}-\left(\Sigma_{11}^{\frac{1}{2}}\int_0^1 w_1(t)dw_1'(t)\Sigma_{11}^{\frac{1}{2}}+\Sigma_{11}\right)\Sigma_{11}^{-1}\Sigma_{12}\right)\\
&= \mathcal{L}\left(U_2\right).
\end{aligned}$$

3.4 We have only to show that $m_1(\theta)=\phi_1(-i\theta)$, where $m_1(\theta)$ is given below (11.40). Girsanov's theorem yields

$$m_1(\theta)=\exp\left\{\frac{\theta}{2}(ab-2d)-\frac{\gamma}{2}\right\}E\left[\exp\left\{\frac{1}{2}(\gamma-ab\theta)Z^2(1)\right\}\right],$$

where $\gamma=\sqrt{-a^2\theta\left(2x+c^2\theta\right)}$ and $dZ(t)=-\gamma Z(t)dt+dw_1(t)$ with $Z(0)=0$. Since $Z(1)\sim \mathrm{N}\left(0\,,(1-e^{-2\gamma})/\left(2\gamma\right)\right)$, we can easily obtain the conclusion.

3.5 We compute the right side of (S19), where X_{OLS} has

$$\psi_1(\theta_1,\,-\theta_2)=\exp\left\{\frac{\theta_1}{2}(2d-ab)\right\}\left[\cos\mu-ab\theta_1\frac{\sin\mu}{\mu}\right]^{-\frac{1}{2}}, \tag{S21}$$

$$\mu=\sqrt{a^2\left(c^2\theta_1^2-2\theta_2\right)}.$$

Then we obtain, for instance,

$$\begin{aligned}
E\left(X_{OLS}\right) &= \int_0^\infty \left.\frac{\partial\psi_1(\theta_1,-\theta_2)}{\partial\theta_1}\right|_{\theta_1=0} d\theta_2\\
&= \frac{\Sigma_{12}}{2}\int_0^\infty\left(\cosh a\sqrt{2\theta}\right)^{-\frac{1}{2}}\left(1+\frac{1}{\cosh a\sqrt{2\theta}}\frac{\sinh a\sqrt{2\theta}}{a\sqrt{2\theta}}\right)d\theta\\
&= \frac{\Sigma_{12}}{2\Sigma_{11}}\left[\int_0^\infty\frac{u}{\sqrt{\cosh u}}du+\int_0^\infty(\cosh u)^{-\frac{3}{2}}\sinh u\,du\right]\\
&= \frac{\Sigma_{12}}{2\Sigma_{11}}\left(c_1+2\right).
\end{aligned}$$

We can compute $E\left(X_{OLS}^2\right)$ and moments of X_{2SLS} and X_{ML} similarly.

3.6 It follows from (11.34) and (11.42) that

$$Y_2 \,\Big|\, \text{vec}\,(Y_1') \sim \text{N}\left(\delta e + Y_1\beta + \Delta Y_1 \Sigma_{11}^{-1}\Sigma_{12},\, \Sigma_{22\cdot 1} I_T\right) .$$

Since $f\left(\text{vec}\,(Y_1')\,,\, Y_2\right) = f_1\left(\text{vec}\,(Y_1')\right) f_2\left(Y_2 \,|\, \text{vec}\,(Y_1')\right)$, the MLE of β is the ordinary LSE of β obtained from $Y_2 = \delta e + Y_1\beta + \Delta Y_1 \gamma + v_2$, where $\gamma = \Sigma_{11}^{-1}\Sigma_{12}$ and $v_2 = \Xi_2 - \Xi_1\gamma$. This gives us $\tilde{\beta}_{ML}$ in (11.45).

3.7 Let us consider first

$$T\left(\tilde{\beta}_{OLS} - \beta\right) = \left(\frac{1}{T^2} Y_1' M Y_1\right)^{-1} \frac{1}{T} Y_1' M \Xi_2\,,$$

where it holds that

$$\begin{aligned} &\mathcal{L}\left(\frac{1}{T^2} Y_1' M Y_1\,,\, \frac{1}{T} Y_1' M \Xi_2\right) \\ &\quad = \quad \mathcal{L}\left(\frac{1}{T^2}\sum_{j=1}^{T}\left(y_{1j} - \bar{y}_1\right)\left(y_{1j} - \bar{y}_1\right)'\,,\, \frac{1}{T}\sum_{j=1}^{T}\left(y_{1,\,j-1} - \bar{y}_1 + \xi_{1j}\right)\xi_{2j}\right) \\ &\quad \longrightarrow \quad \mathcal{L}\left(\tilde{V},\, \tilde{U}_1 + \tilde{U}_2 + \Sigma_{12}\right) . \end{aligned}$$

Thus it follows that $\mathcal{L}\left(T\left(\tilde{\beta}_{OLS} - \beta\right)\right) \longrightarrow \mathcal{L}\left(\tilde{V}^{-1}\left(\tilde{U}_1 + \tilde{U}_2 + \Sigma_{12}\right)\right)$. We consider next

$$T\left(\tilde{\beta}_{2SLS} - \beta\right) = \left(\frac{1}{T^2} Y_1' \tilde{P} M \tilde{P} Y_1\right)^{-1} \frac{1}{T} Y_1' \tilde{P} M \Xi_2\,,$$

where $\tilde{Y}_1 = \tilde{P} Y_1$. Since $(e\,, Y_{-1})'\,\tilde{P} = (e\,, Y_{-1})'$, we find

$$\begin{aligned} \frac{1}{T^2} Y_1' \tilde{P} M \tilde{P} Y_1 &= \frac{1}{T^2} Y_1' M Y_1 + o_p(1)\,, \\ \frac{1}{T} Y_1' \tilde{P} M \Xi_2 &= \frac{1}{T} Y_{-1}' \Xi_2 - \frac{1}{T} Y_1' e e' \Xi_2 + o_p(1) \end{aligned}$$

so that $\mathcal{L}\left(T\left(\tilde{\beta}_{2SLS} - \beta\right)\right) \longrightarrow \mathcal{L}\left(\tilde{V}^{-1}\left(\tilde{U}_1 + \tilde{U}_2\right)\right)$. Finally we consider

$$T\left(\tilde{\beta}_{ML} - \beta\right) = \left(\frac{1}{T^2} Y_1' M_2 Y_1\right)^{-1} \frac{1}{T} Y_1' M_2 \Xi \kappa\,,$$

where $\Xi = (\Xi_1, \Xi_2)$, $\kappa = (-\gamma', 1)'$ with $\gamma = \Sigma_{11}^{-1}\Sigma_{12}$ and

$$\begin{aligned}
\frac{1}{T^2}Y_1'M_2Y_1 &= \frac{1}{T^2}Y_1'MY_1 + o_p(1)\,, \\
\frac{1}{T}Y_1'M_2\Xi &= \frac{1}{T}Y_1'\Xi - \frac{1}{T}\left(Y_1'e\,, Y_1'\Xi_1\right)\begin{pmatrix} e'e & e'\Xi_1 \\ \Xi_1'e & \Xi_1'\Xi_1 \end{pmatrix}^{-1}\begin{pmatrix} e'\Xi \\ \Xi_1'\Xi \end{pmatrix} \\
&= \frac{1}{T}Y_1'\Xi - \frac{1}{T^2}Y_1'ee'\Xi - \frac{1}{T}Y_1'\Xi_1\left(\Xi_1'\Xi_1\right)^{-1}\Xi_1'\Xi + o_p(1)\,.
\end{aligned}$$

Since $Y_1'M_2\Xi\kappa/T$ converges in distribution to

$$\begin{aligned}
&\left[\Sigma_{11}^{\frac{1}{2}}\int_0^1 w_1(t)dw'(t)B' + (\Sigma_{11}\,, \Sigma_{12}) - \Sigma_{11}^{\frac{1}{2}}\int_0^1 w_1(t)dtw'(1)B'\right. \\
&\quad \left. - \left(\Sigma_{11}^{\frac{1}{2}}\int_0^1 w_1(t)dw_1'(t)\Sigma_{11}^{\frac{1}{2}} + \Sigma_{11}\right)\Sigma_{11}^{-1}\left(\Sigma_{11}\,, \Sigma_{12}\right)\right]\begin{pmatrix} -\gamma \\ 1 \end{pmatrix} \\
&\quad = \tilde{U}_2\,,
\end{aligned}$$

where B is defined in (11.35), we can deduce that $\mathcal{L}\left(T\left(\tilde{\beta}_{ML} - \beta\right)\right) \longrightarrow \mathcal{L}\left(\tilde{V}^{-1}\tilde{U}_2\right)$.

3.8 Let us compute

$$\begin{aligned}
m_2(\theta) &= E\left[E\left\{e^{\theta X_2}\middle| w_1\right\}\right] \\
&= E\left[\exp\left\{\theta E\left(X_2|w_1\right) + \frac{\theta^2}{2}V\left(X_2|w_1\right)\right\}\right] \\
&= \exp\left\{\frac{\theta}{2}(ab - 2d)\right\}E\left[\exp\left\{c_1\int_0^1 \tilde{w}_1^2(t)dt\right.\right. \\
&\qquad\qquad \left.\left. + c_2w_1^2(1) + c_3w_1(1)\int_0^1 w_1(t)dt\right\}\right]\,,
\end{aligned}$$

where

$$c_1 = a^2\theta\left(x + \frac{c^2\theta}{2}\right)\,, \qquad c_2 = -\frac{ab\theta}{2}\,, \qquad c_3 = ab\theta\,.$$

Girsanov's theorem leads us to

$$\begin{aligned}
m_2(\theta) &= \exp\left\{\frac{\theta}{2}(ab - 2d) - \frac{\gamma}{2}\right\} \\
&\quad \times E\left[\exp\left\{-c_1\left(\int_0^1 Z(t)dt\right)^2 + \left(c_2 + \frac{\gamma}{2}\right)Z^2(1) + c_3Z(1)\int_0^1 Z(t)dt\right\}\right]\,,
\end{aligned}$$

where $dZ(t) = -\gamma Z(t)dt + dw_1(t)$ with $\gamma = \sqrt{-2c_1}$ and $Z(0) = 0$. We obtain $m_2(\theta) = \phi_2(-i\theta)$ after some algebra.

3.9 We compute the right side of (S19), where Y_{OLS} has

$$
\begin{aligned}
\text{(S22)}\quad \psi_1(\theta_1\,,-\theta_2) &= \exp\left\{\frac{\theta_1}{2}(2d-ab)\right\}\left[\frac{2a^2b^2\theta_1^2}{\mu^4}(\cos\mu-1)+\left(1+\frac{a^2b^2\theta_1^2}{\mu^2}\right)\frac{\sin\mu}{\mu}\right]^{-\frac{1}{2}}, \\
\mu &= \sqrt{a^2\left(c^2\theta_1^2-2\theta_2\right)}\,.
\end{aligned}
$$

Proceeding in the same way as in the solution to Problem 3.5 we obtain moments of $Y_{OLS}\,, Y_{2SLS}$ and Y_{ML}.

4.1 It follows from (11.47) that

$$
T\left(\hat{\beta}_{OLS}-\beta\right)=\left(\frac{1}{T^2}\sum_{j=1}^{T}y_{1j}y_{1j}'\right)^{-1}\frac{1}{T}\sum_{j=1}^{T}y_{1j}g'(L)\varepsilon_j\,.
$$

Using the weak convergence results on the auxiliary process $\{z_j\}$ introduced below (11.47) we can deduce that

$$
\mathcal{L}\left(\frac{1}{T^2}\sum_{j=1}^{T}y_{1j}y_{1j}'\,,\ \frac{1}{T}\sum_{j=1}^{T}y_{1j}g'(L)\varepsilon_j\right)\longrightarrow\mathcal{L}\left(R\,,\ Q_1+Q_2+\Lambda_{12}\right)\,,
$$

which establishes the theorem.

4.2 We compute the right side of (S19) when $\psi_1(\theta_1\,,-\theta_2)$ is given by (S21), where a, b, c and d are defined in (11.49). We obtain, for instance,

$$
\left.\frac{\partial\psi_1(\theta_1\,,-\theta_2)}{\partial\theta_1}\right|_{\theta_1=0}=\frac{\left(\cosh a\sqrt{2\theta_2}\right)^{-\frac{1}{2}}}{2}\left(2d-ab+\frac{ab}{\cosh a\sqrt{2\theta_2}}\frac{\sinh a\sqrt{2\theta_2}}{a\sqrt{2\theta_2}}\right)\,,
$$

which yields $E\left(X_{OLS}\right)$. We can compute $E\left(X_{OLS}^2\right)$ similarly.

4.3 It follows from (11.50) that $T\left(\hat{\beta}_{FM}-\beta\right)=V_T^{-1}U_T$, where

$$
\begin{aligned}
U_T &= \frac{1}{T}\sum_{j=1}^{T}y_{1j}\left(g'(L)\varepsilon_j-\hat{\Omega}_{21}\hat{\Omega}_{11}^{-1}\Phi_1'(L)\varepsilon_j\right)-\hat{\Lambda}_{12}+\hat{\Lambda}_{11}\hat{\Omega}_{11}^{-1}\hat{\Omega}_{12}\,, \\
V_T &= \frac{1}{T^2}\sum_{j=1}^{T}y_{1j}y_{1j}'\,.
\end{aligned}
$$

Because of the weak convergence results on the auxiliary process $\{z_j\}$ and Theorem 11.7 we can deduce that

$$\begin{aligned}
\mathcal{L}\left(U_T\,,V_T\right) \quad &\longrightarrow \quad \mathcal{L}\Big(Q_1+Q_2+\Lambda_{12}-\Big(\Omega_{11}^{\frac{1}{2}}\int_0^1 w_1(t)dw_1'(t)\Omega_{11}^{\frac{1}{2}}+\Lambda_{11}\Big)\,\Omega_{11}^{-1}\Omega_{12} \\
&\qquad\qquad -\Lambda_{12}+\Lambda_{11}\Omega_{11}^{-1}\Omega_{12}\,,R\Big) \\
&= \quad \mathcal{L}\left(Q_2\,,R\right)\,,
\end{aligned}$$

which establishes the theorem.

4.4 Let us consider $T\left(\tilde{\beta}_{OLS}-\beta\right)=\tilde{V}_T^{-1}\tilde{U}_{1T}$ and $T\left(\tilde{\beta}_{FM}-\beta\right)=\tilde{V}_T^{-1}\tilde{U}_{2T}$, where

$$\begin{aligned}
\tilde{U}_{1T} &= \frac{1}{T}\sum_{j=1}^{T}\left(y_{1j}-\bar{y}_1\right)g'\left(L\right)\varepsilon_j\,, \\
\tilde{U}_{2T} &= \frac{1}{T}\sum_{j=1}^{T}\left(y_{1j}-\bar{y}_1\right)\Big(g'\left(L\right)\varepsilon_j-\tilde{\Omega}_{21}\tilde{\Omega}_{11}^{-1}\Phi_1'\left(L\right)\varepsilon_j\Big)-\tilde{\Lambda}_{12}+\tilde{\Lambda}_{11}\tilde{\Omega}_{11}^{-1}\tilde{\Omega}_{12}\,, \\
\tilde{V}_T &= \frac{1}{T^2}\sum_{j=1}^{T}\left(y_{1j}-\bar{y}_1\right)\left(y_{1j}-\bar{y}_1\right)'\,.
\end{aligned}$$

If we construct the auxiliary process :

$$\Delta z_j=\begin{pmatrix}\Delta y_{1j}\\ \\ \Delta x_j\end{pmatrix}=\begin{pmatrix}\Phi_1'(L)\\ \\ g'(L)\end{pmatrix}\varepsilon_j\,,\qquad z_0=0\,,$$

we have that

$$\mathcal{L}\left(\frac{1}{T^2}\sum_{j=1}^{T}\left(z_j-\bar{z}\right)\left(z_j-\bar{z}\right)'\right)\longrightarrow\mathcal{L}\left(D\int_0^1\tilde{w}(t)\tilde{w}'(t)dtD'\right)\,,$$

$$\mathcal{L}\left(\frac{1}{T}\sum_{j=1}^{T}\left(z_j-\bar{z}\right)\Delta z_j'\right)\longrightarrow\mathcal{L}\left(D\int_0^1\tilde{w}(t)dw'(t)D'+\Lambda\right)\,.$$

Then it is easy to deduce that

$$\begin{aligned}
\mathcal{L}\left(\tilde{U}_{1T}\,,\tilde{V}_T\right) \quad &\longrightarrow \quad \mathcal{L}\left(\tilde{Q}_1+\tilde{Q}_2+\Lambda_{12}\,,\tilde{R}\right)\,, \\
\mathcal{L}\left(\tilde{U}_{2T}\,,\tilde{V}_T\right) \quad &\longrightarrow \quad \mathcal{L}\Big(\tilde{Q}_1+\tilde{Q}_2+\Lambda_{12}-\Big(\Omega_{11}^{\frac{1}{2}}\int_0^1\tilde{w}_1(t)dw_1'(t)\Omega_{11}^{\frac{1}{2}}+\Lambda_{11}\Big) \\
&\qquad\qquad \times\Omega_{11}^{-1}\Omega_{12}-\Lambda_{12}+\Lambda_{11}\Omega_{11}^{-1}\Omega_{12}\,,\,\tilde{R}\Big) \\
&= \quad \mathcal{L}\left(\tilde{Q}_2\,,\tilde{R}\right)\,.
\end{aligned}$$

Thus the theorem is established.

4.5 We compute the right side of (S19) when $\psi_1(\theta_1, -\theta_2)$ is given by (S22), where a, b, c and d are defined in (11.49). We obtain, for instance,

$$\left.\frac{\partial\psi_1(\theta_1, -\theta_2)}{\partial\theta_1}\right|_{\theta_1=0} = \frac{2d-ab}{2}\frac{\sqrt{a\sqrt{2\theta_2}}}{\sqrt{\sinh a\sqrt{2\theta_2}}}$$

so that

$$E(Y_{OLS}) = \frac{2d-ab}{2a^2}\int_0^\infty \frac{u^{\frac{3}{2}}}{\sqrt{\sinh u}}du = \frac{2d-ab}{2a^2}d_1 \, .$$

We can compute $E(Y_{OLS}^2)$ similarly.

6.1 It follows from the text that

$$\mathcal{L}(T(\hat{\rho}-1)) \longrightarrow \mathcal{L}\left(\frac{(-X_1', 1)\,B\int_0^1 w(t)dw'(t)B'\begin{pmatrix}-X_1\\1\end{pmatrix}}{(-X_1', 1)\,B\int_0^1 w(t)w'(t)dtB'\begin{pmatrix}-X_1\\1\end{pmatrix}} + R\right).$$

Since we have

$$\begin{aligned}(-X_1', 1)\,Bw(t) &= \left(-\int_0^1 (B_2'w_1(t)+B_3w_2(t))\,w_1'(t)dtB_1\right.\\ &\quad \left.\times\left(B_1\int_0^1 w_1(t)w_1'(t)dtB_1\right)^{-1}, 1\right)\begin{pmatrix}B_1w_1(t)\\ \\ B_2'w_1(t)+B_3w_2(t)\end{pmatrix}\\ &= B_3Q(t)\,,\end{aligned}$$

we can prove (11.62).

6.2 Consider

$$\frac{1}{T}\hat{Z}_\rho = \hat{\rho} - 1 - \left(\hat{\sigma}_L^2 - \hat{\sigma}_S^2\right)\Big/\left(\frac{2}{T}\sum_{j=2}^T \hat{\eta}_{j-1}^2\right).$$

Noting that

$$\hat{\eta}_j = y_{2j} - \hat{\beta}'y_{1j} = g'(L)\,\varepsilon_j - \left(\hat{\beta}-\beta\right)'y_{1j}\,,$$

we obtain

$$\frac{1}{T}\sum_{j=2}^{T}\hat{\eta}_{j-1}^2 \longrightarrow \gamma(0)\,, \qquad \frac{1}{T}\sum_{j=2}^{T}\hat{\eta}_{j-1}\hat{\eta}_j \longrightarrow \gamma(1)$$

in probability so that $\operatorname{plim}\hat{\rho} = \gamma(1)/\gamma(0) = \rho$. Since

$$\hat{\eta}_j - \hat{\rho}\hat{\eta}_{j-1} = g'(L)\,\varepsilon_j - \rho g'(L)\,\varepsilon_{j-1} - \left(\hat{\beta}-\beta\right)'(y_{1,j} - \rho y_{1,j-1})\ ,$$

we have

$$\begin{aligned}\hat{\sigma}_S^2 &= \frac{1}{T}\sum_{j=2}^{T}(\hat{\eta}_j - \hat{\rho}\hat{\eta}_{j-1})^2 \longrightarrow E\left\{(g'(L)\,\varepsilon_j - \rho g'(L)\,\varepsilon_{j-1})^2\right\}\\ &= \left(\gamma^2(0) - \gamma^2(1)\right)\Big/\,\gamma(0)\,.\end{aligned}$$

The estimator $\hat{\sigma}_L^2$ converges in probability to 2π times the spectrum of $\{g'(L)\,\varepsilon_j - \rho g'(L)\,\varepsilon_{j-1}\}$ evaluated at the origin, that is

$$\operatorname{plim}\hat{\sigma}_L^2 = (g' - \rho g')(g - \rho g) = (\gamma(0) - \gamma(1))^2\, g'g\Big/\,\gamma^2(0)\,.$$

Therefore it follows that

$$\begin{aligned}\operatorname{plim}\left(\frac{1}{T}\hat{Z}_\rho\right) &= \frac{\gamma(1)}{\gamma(0)} - 1 - \frac{1}{2\gamma(0)}\left[\frac{(\gamma(0)-\gamma(1))^2}{\gamma^2(0)}g'g - \frac{\gamma^2(0)-\gamma^2(1)}{\gamma(0)}\right]\\ &= -\frac{(\gamma(0)-\gamma(1))^2}{2\gamma^2(0)}\left(1 + \frac{g'g}{\gamma(0)}\right)\,.\end{aligned}$$

6.3 We can deduce from the arguments leading to (11.63) that

$$\mathcal{L}\left(\frac{1}{T^2}\sum_{j=2}^{T}\hat{\eta}_{j-1}^2\,, \hat{\sigma}_L^2\right) \longrightarrow \mathcal{L}\left(B_3^2\int_0^1 Q^2(t)dt\,, B_3^2 S'S\right)\,,$$

which proves (11.66) by the continuous mapping theorem. Under H_1 we have

$$\begin{aligned}\frac{1}{\sqrt{T}}\hat{Z}_t &= \left(\frac{1}{\hat{\sigma}_L^2}\frac{1}{T}\sum_{j=2}^{T}\hat{\eta}_{j-1}^2\right)^{\frac{1}{2}}\frac{1}{T}\hat{Z}_\rho\\ &\longrightarrow \sqrt{\frac{\gamma^3(0)}{(\gamma(0)-\gamma(1))^2\, g'g}}\left(-\frac{(\gamma(0)-\gamma(1))^2}{2\gamma^2(0)}\right)\left(1+\frac{g'g}{\gamma(0)}\right)\\ &= -\frac{\gamma(0)-\gamma(1)}{2\sqrt{\gamma(0)g'g}}\left(1+\frac{g'g}{\gamma(0)}\right) \qquad \text{in probability}\,.\end{aligned}$$

6.4 Let us consider

$$
\begin{aligned}
\hat{v}_j &= y_{2j} - \hat{\beta}'_{FM} y_{1j} - \hat{\Omega}_{21}\hat{\Omega}_{11}^{-1}\Delta y_{1j} \\
&= \frac{c}{T}\frac{\xi_{2j}}{1-L} + \gamma(L)\,\xi_{2j} - \left(\hat{\beta}_{FM} - \beta\right)' y_{1j} - \hat{\Omega}_{21}\hat{\Omega}_{11}^{-1} G(L)\,\xi_{1j}\,,
\end{aligned}
$$

where

$$
\begin{aligned}
T\left(\hat{\beta}_{FM} - \beta\right) &= \left(\frac{1}{T^2}\sum_{j=1}^{T} y_{1j}y'_{1j}\right)^{-1}\Bigg[\frac{1}{T}\sum_{j=1}^{T} y_{1j} \\
&\quad \times \left(\frac{c}{T}\frac{\xi_{2j}}{1-L} + \gamma(L)\,\xi_{2j} - \hat{\Omega}_{21}\hat{\Omega}_{11}^{-1}G(L)\,\xi_{1j}\right) - \hat{\Lambda}_{12} + \hat{\Lambda}_{11}\hat{\Omega}_{11}^{-1}\hat{\Omega}_{12}\Bigg]\,.
\end{aligned}
$$

Defining the auxiliary process $\{z_j\}$ as in the text, we deduce that $\mathcal{L}\left(T\left(\hat{\beta}_{FM} - \beta\right)\right) \longrightarrow \mathcal{L}\left(\left(J'_1\right)^{-1}Y\right)$, where

$$
\begin{aligned}
Y &= J_3\left(\int_0^1 w_1(t)w'_1(t)dt\right)^{-1}\int_0^1 w_1(t)dw_2(t) \\
&\quad + \frac{c}{\gamma(1)}\left(\int_0^1 w_1(t)w'_1(t)dt\right)^{-1}\int_0^1 w_1(t)w'(t)dt\begin{pmatrix} J_2 \\ J_3 \end{pmatrix}\,.
\end{aligned}
$$

Then it follows that

$$
\begin{aligned}
\mathcal{L}\left(\frac{1}{T}\left(\sum_{j=1}^{T}\hat{v}_j\right)^2\right) &\longrightarrow \mathcal{L}\Bigg(\left(J'_2\,, J_3\right)w(1) - Y'\int_0^1 w_1(t)dt - J'_2 w_1(1) \\
&\qquad + \frac{c}{\gamma(1)}\left(J'_2\,, J_3\right)w(1)\Bigg)^2 \\
&= \mathcal{L}\left(J_3^2\left(Y_1 + cY_2\right)^2\right)\,.
\end{aligned}
$$

Since $\operatorname{plim}\hat{\Omega}_{22\cdot 1} = \Omega_{22\cdot 1} = J_3^2$, the theorem is established.

6.5 Under the fixed alternative we have that $\{\hat{v}_j\}$ is $I(1)$ so that $\sum_{j=1}^{T}\hat{v}_j = O_p\left(T\sqrt{T}\right)$. Since $\hat{\Omega}_{22\cdot 1}$ is constructed from the long-run variance of $\left\{y_{2j} - \hat{\beta}'_{FM}y_{1j}\right\}$, it holds that $\hat{\Omega}_{22\cdot 1} = O_p(T)$. Thus $\hat{S}_{T1} = O_p(T)$.

6.6 The present model may be expressed as

(S23) $$v = \delta e + (\kappa C + I_T)\,\Xi_2 \sim \mathrm{N}\left(\delta e\,, \sigma^2\Omega(\kappa)\right)\,,$$

where $e = (1, \cdots, 1)' : T \times 1$, $\Xi_2 = (\xi_{21}, \cdots, \xi_{2T})'$, and $\Omega(\kappa) = (\kappa C + I_T)(\kappa C' + I_T)$ with C being the random walk generating matrix. Let $L(\kappa, \delta, \sigma^2)$ be the log-likelihood for v. Then we have

$$L\left(\kappa, \delta, \sigma^2\right) = -\frac{T}{2}\log\left(2\pi\sigma^2\right) - \frac{1}{2}\log|\Omega(\kappa)| - \frac{1}{2\sigma^2}(v - \delta e)'\Omega^{-1}(\kappa)(v - \delta e)\ .$$

It is easy to obtain

$$\left.\frac{\partial^2 L}{\partial\kappa^2}\right|_{H_0} = \text{constant} + T\frac{v'MCC'Mv}{v'Mv}\,,$$

where $M = I_T - ee'/T$. This gives us the LBIU statistic S_{T2}.

6.7 Consider

$$S_{T2} = \frac{1}{T^2}v'MCC'Mv \Big/ \frac{1}{T}v'Mv\,,$$

where v is defined in (S23) with κ replaced by c/T. We have

$$\begin{aligned}\frac{1}{T}v'Mv &= \frac{1}{T}\Xi_2'\left(\frac{c}{T}C' + I_T\right)M\left(\frac{c}{T}C + I_T\right)\Xi_2\\ &\longrightarrow \sigma^2 \qquad \text{in probability}\,.\end{aligned}$$

Moreover it follows from Theorem 5.13 that

$$\begin{aligned}\mathcal{L}\left(\frac{1}{T^2\sigma^2}v'MCC'Mv\right) &= \mathcal{L}\left(\frac{1}{T^2\sigma^2}\Xi_2'\left(C'MC + \frac{c^2}{T^2}\left(C'MC\right)^2\right)\Xi_2 + o_p(1)\right)\\ &\longrightarrow \mathcal{L}\left(\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n} + \frac{c^2}{\lambda_n^2}\right)Z_n^2\right)\,,\end{aligned}$$

where $\{\lambda_n\}$ is a sequence of eigenvalues of the kernel $K(s,t) = \min(s,t) - st$. Since $\lambda_n = n^2\pi^2$ in the present case, (11.75) is established.

6.8 We first note that

$$\tilde{v}_j - \bar{\tilde{v}} = \frac{c}{T}\frac{\xi_{2j} - \bar{\xi}_2}{1 - L} + \gamma(L)\left(\xi_{2j} - \bar{\xi}_2\right) - \left(\tilde{\beta}_{FM} - \beta\right)'(y_{1j} - \bar{y}_1) - \tilde{\Omega}_{21}\tilde{\Omega}_{11}^{-1}(\Delta y_{1j} - \Delta\bar{y}_1)\ ,$$

where

$$\begin{aligned}T\left(\tilde{\beta}_{FM} - \beta\right) &= \left(\frac{1}{T^2}\sum_{j=1}^{T}(y_{1j} - \bar{y}_1)(y_{1j} - \bar{y}_1)'\right)^{-1}\left[\frac{1}{T}\sum_{j=1}^{T}(y_{1j} - \bar{y}_1)\right.\\ &\qquad \left.\times\left(\frac{c}{T}\frac{\xi_{2j}}{1 - L} + \gamma(L)\xi_{2j} - \tilde{\Omega}_{21}\tilde{\Omega}_{11}^{-1}\Delta y_{1j}\right) - \tilde{\Lambda}_{12} + \tilde{\Lambda}_{11}\tilde{\Omega}_{11}^{-1}\tilde{\Omega}_{12}\right]\,.\end{aligned}$$

We can deduce that $\mathcal{L}\left(T\left(\tilde{\beta}_{FM}-\beta\right)\right) \longrightarrow \mathcal{L}\left((J_1')^{-1} Z\right)$, where

$$\begin{aligned} Z &= J_3\left(\int_0^1 \tilde{w}_1(t)\tilde{w}_1'(t)dt\right)^{-1}\int_0^1 \tilde{w}_1(t)dw_2(t) \\ &\quad +\frac{c}{\gamma(1)}\left(\int_0^1 \tilde{w}_1(t)\tilde{w}_1'(t)dt\right)^{-1}\int_0^1 \tilde{w}_1(t)\tilde{w}'(t)dt\begin{pmatrix} J_2 \\ J_3 \end{pmatrix}. \end{aligned}$$

Defining the partial sum process :

$$X_T(t) = \frac{1}{\sqrt{T}}\sum_{j=1}^{[Tt]}(\tilde{v}_j - \bar{\tilde{v}}) + (Tt-[Tt])\frac{1}{\sqrt{T}}\left(\tilde{v}_{[Tt]+1}-\bar{\tilde{v}}\right),$$

we obtain $\mathcal{L}(X_T) \longrightarrow \mathcal{L}(X)$, where

$$\begin{aligned} X(t) &= (J_2', J_3)(w(t)-tw(1)) - Z'\int_0^t \tilde{w}_1(s)ds - J_2'(w_1(t)-tw_1(1)) \\ &\quad +\frac{c}{\gamma(1)}(J_2', J_3)\int_0^t \tilde{w}(s)ds \\ &= J_3(Z_1(t)+cZ_2(t)). \end{aligned}$$

Then it follows that

$$\begin{aligned} \mathcal{L}\left(\tilde{S}_{T2}\right) &= \mathcal{L}\left(\frac{1}{T}\sum_{j=1}^{T} X_T^2\left(\frac{j}{T}\right)\Big/\tilde{\Omega}_{22\cdot 1}\right) \\ &\longrightarrow \mathcal{L}\left(\int_0^1 X^2(t)dt\Big/J_3^2\right) = \mathcal{L}\left(\int_0^1 (Z_1(t)+cZ_2(t))^2\,dt\right). \end{aligned}$$

8.1 It follows from (11.97) and (11.98) that

$$\mathcal{L}\left(\frac{1}{T^{2d}}Y_1'Y_1, \frac{1}{T^d}Y_1'\Xi_2\right) \longrightarrow \mathcal{L}(V, U_1+U_2).$$

Thus $\mathcal{L}\left(T^d\left(\hat{\beta}_{OLS}-\beta\right)\right) \longrightarrow \mathcal{L}\left(V^{-1}(U_1+U_2)\right)$. Since

$$\frac{1}{T^{2d}}Y_1'P_{-d}Y_1 = \frac{1}{Y^{2d}}Y_1'Y_1 + o_p(1), \qquad \frac{1}{T^d}Y_1'P_{-d}\Xi_2 = \frac{1}{Y^d}Y_1'\Xi_2 + o_p(1),$$

we also have $\mathcal{L}\left(T^d\left(\hat{\beta}_{2SLS}-\beta\right)\right) \longrightarrow \mathcal{L}\left(V^{-1}(U_1+U_2)\right)$. Noting that

$$\begin{aligned} \frac{1}{T^{2d}}Y_1'M_dY_1 &= \frac{1}{T^{2d}}Y_1'Y_1 + o_p(1), \\ \frac{1}{T^d}Y_1'M_d\Xi_2 &= \frac{1}{T^d}Y_1'\Xi_2 - \frac{1}{T^d}Y_1'\Xi_1\left(\frac{1}{T}\Xi_1'\Xi_1\right)^{-1}\frac{1}{T}\Xi_1'\Xi_2, \end{aligned}$$

we deduce that $\mathcal{L}\left(T^d\left(\hat{\beta}_{ML}-\beta\right)\right)\longrightarrow\mathcal{L}\left(V^{-1}U_2\right)$.

8.2 Let us put

$$1-L^4=\prod_{k=1}^{4}\left(1-\frac{1}{\theta_k}L\right)\,,$$

where $\theta_1=1\,,\theta_2=-1\,,\theta_3=i$ and $\theta_4=-i$. Then it follows from (11.101) that

$$\begin{aligned}\Phi(L) &= A_1\left(1+L+L^2+L^3\right)+A_2\left(1-L+L^2-L^3\right)\\ &\quad +A_3\left(1-L^2\right)(1-iL)+A_4\left(1-L^2\right)(1+iL)+\left(1-L^4\right)\tilde{\Phi}(L)\ ,\end{aligned}$$

where A_4 must be the complex conjugate of A_3 since the coefficients of $\Phi(L)$ are real. Thus we may put $A_3=G+iH$ and $A_4=G-iH$ with G and H being real, which leads us to the expansion in (11.102).

8.3 Using (11.101) we can expand $\alpha_3(L)$ as

$$\begin{aligned}\alpha_3(L) &= \gamma_1(1-iL)+\gamma_2(1+iL)+\left(1+L^2\right)\tilde{\alpha}_3(L)\\ &= \alpha_{30}+\alpha_{31}L+\left(1+L^2\right)\tilde{\alpha}_3(L)\ .\end{aligned}$$

Then it is seen that $\alpha_3'(i)\,\Phi(i)=0'$ is equivalent to $(\alpha_{30}+\alpha_{31}i)'\,\Phi(i)=0'$.

8.4 The fact that $\mathcal{L}\left(\mathrm{N}\left(\hat{\beta}_{OLS}-\beta\right)\right)\longrightarrow\mathcal{L}\left(V^{-1}\left(U_1+U_2+m\Sigma_{12}\right)\right)$ comes from the continuous mapping theorem and the remark described above the present theorem. Since

$$\frac{1}{N^2}Y_1'P_{-m}Y_1=\frac{1}{N^2}Y_1'Y_1+o_p(1)\,,\qquad \frac{1}{N}Y_1'P_{-m}\Xi_2=\frac{1}{N}Y_{-m}'\Xi_2+o_p(1)\,,$$

we obtain $\mathcal{L}\left(\mathrm{N}\left(\hat{\beta}_{2SLS}-\beta\right)\right)\longrightarrow\mathcal{L}\left(V^{-1}\left(U_1+U_2\right)\right)$. Noting that

$$\begin{aligned}\frac{1}{N^2}Y_1'M_mY_1 &= \frac{1}{N^2}Y_1'Y_1+o_p(1)\,,\\ \frac{1}{N}Y_1'M_m\Xi_2 &= \frac{1}{N}Y_1'\Xi_2-\frac{1}{N}Y_1'\Xi_1\left(\frac{1}{N}\Xi_1'\Xi_1\right)^{-1}\frac{1}{N}\Xi_1'\Xi_2\,,\end{aligned}$$

we deduce that

$$\mathcal{L}\left(\frac{1}{N^2}Y_1'M_mY_1\,,\,\frac{1}{N}Y_1'M_m\Xi_2\right)\longrightarrow\mathcal{L}\left(V,\,U_2\right)\ .$$

Thus we have $\mathcal{L}\left(\mathrm{N}\left(\hat{\beta}_{ML}-\beta\right)\right) \longrightarrow \mathcal{L}\left(V^{-1}U_2\right)$.

8.5 Given $\underset{\sim}{w}_1 = \left\{\underset{\sim}{w}_1(t)\right\}$ the quantity $X_{OLS}(m)$ is normal with

$$
\begin{aligned}
E\left(X_{OLS}(m) \mid \underset{\sim}{w}_1\right) &= a^2x\int_0^1 \underset{\sim}{w}_1'(t)\,\underset{\sim}{w}_1(t)dt - \frac{ab}{2}\,\underset{\sim}{w}_1'(1)\,\underset{\sim}{w}_1(1) + \frac{m}{2}\left(ab-2d\right)\,, \\
V\left(X_{OLS}(m) \mid \underset{\sim}{w}_1\right) &= a^2c^2\int_0^1 \underset{\sim}{w}_1'(t)\,\underset{\sim}{w}_1(t)dt\,.
\end{aligned}
$$

Then we obtain $E\left\{\exp\left(i\theta X_{OLS}(m)\right)\right\} = \left\{\phi_1(\theta)\right\}^m$, which proves the theorem.