Time Series Analysis

Nonstationary and Noninvertible Distribution Theory

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Chapter 4 The Stochastic Process Approach

We present a method for computing the c.f.'s of quadratic or bilinear functionals of the Brownian motion, where functionals involve the single Riemann integral or the Ito integral. In doing so we use a theorem concerning a transformation of measures induced by stochastic processes, from which the present approach called the *stochastic process approach* originates. It is recognized that the stochastic process approach does not require the knowledge of eigenvalues, unlike the eigenvalue approach presented in Chapter 1. Advantages and disadvantages of the present approach are discussed in the last section.

4.1. Girsanov's theorem : case 1

We have discussed in Chapter 3 some FCLT's and have indicated how to derive weak convergence results for various statistics. It is seen that the limiting random variables are functionals of the Brownian motion. In statistical applications we need to compute distribution functions of those limiting random variables. In general, however, it is difficult to derive the distribution functions directly.

Here we present a method that we call the stochastic process approach for computing the c.f.'s or m.g.f.'s for quadratic and bilinear functionals of the Brownian motion. In this and next sections quadratic functionals are dealt with, while bilinear functionals are treated in Section 3. To illustrate the present methodology we reconsider the statistics presented in Chapter 1, the c.f.'s for which were derived by the eigenvalue approach. It should be emphasized that the present approach does not require the knowledge of eigenvalues, unlike the eigenvalue approach.

The stochastic process approach relies on *Girsanov's theorem* concerning a transformation of measures induced by stochastic processes. The idea is as follows. Suppose that $f(X)$ is a functional of a stochastic process $\{X(t)\}$ and that we would like to compute $E(f(X))$. Then, defining an auxiliary process $\{Y(t)\}$ that is equivalent in the sense of measures μ_X and μ_Y the two processes induce, Girsanov's theorem yields $E(f(X)) = E(f(Y)d\mu_X(Y)/d\mu_Y)$, where the expectation on the left is taken with respect to μ_X, while that on the right with respect to μ_Y. An appropriate choice of $\{Y(t)\}$ may make the computation feasible.

As an example let us take up

$$S_1 = \int_0^1 X^2(t)dt\,, \tag{4.1}$$

where $\{X(t)\}$ is the O-U process defined by

$$dX(t) = -\alpha X(t)dt + dw(t) \Leftrightarrow X(t) = e^{-\alpha t}X(0) + e^{-\alpha t}\int_0^t e^{\alpha s}dw(s) \tag{4.2}$$

with $\{w(t)\}$ being the one-dimensional standard Brownian motion. Note that $X(0)$ is assumed to be independent of increments of $\{w(t)\}$.

Our concern is to compute the c.f. $\phi_{S_1}(\theta) = E(e^{i\theta S_1})$. For this purpose the following theorem due to Girsanov (1960) (see, also, Liptser and Shiryayev (1977, p.277)) is useful.

Theorem 4.1. *Let $X = \{X(t) : 0 \le t \le 1\}$ and $Y = \{Y(t) : 0 \le t \le 1\}$ be the O-U processes on $C = C[0,1]$ defined by*

$$(4.3) \qquad dX(t) = -\alpha X(t)dt + dw(t)\,,$$

$$(4.4) \qquad dY(t) = -\beta Y(t)dt + dw(t)\,, \qquad X(0) = Y(0)\,.$$

Let μ_X and μ_Y be probability measures on $(C, \mathcal{B}(C))$ induced by X and Y, respectively, by the relation

$$(4.5) \qquad \mu_X(A) = P(\omega : X \in A)\,, \quad \mu_Y(A) = P(\omega : Y \in A)\,, \quad A \in \mathcal{B}(C)\,.$$

Then measures μ_X and μ_Y are equivalent and

$$(4.6) \qquad \frac{d\mu_X}{d\mu_Y}(x) = \exp\left[(\beta - \alpha)\int_0^1 x(t)dx(t) - \frac{\alpha^2 - \beta^2}{2}\int_0^1 x^2(t)dt\right]\,,$$

where the left side is the Radon-Nikodym derivative evaluated at $x \in C$ with $x(0) = X(0)$.

Roughly speaking, the Radon-Nikodym derivative in (4.6) is an extended version of the likelihood ratio under contiguity. Suppose that $y_j = \rho_n(\alpha)y_{j-1} + \varepsilon_j$ $(j = 1, \cdots, n)$ with $y_0 = 0$ and $\{\varepsilon_j\} \sim \mathrm{NID}(0,1)$. Let $l_n(\alpha)$ be the likelihood for $y_1, \cdots, y_n$ under $\rho_n(\alpha) = 1 - \dfrac{\alpha}{n}$. Then it holds (Problem 1.1) that

$$(4.7) \qquad \mathcal{L}\left(\left.\frac{l_n(\alpha)}{l_n(\beta)}\right|_\gamma\right) \longrightarrow \mathcal{L}\left(\exp\left[(\beta - \alpha)\int_0^1 y(t)dy(t) - \frac{\alpha^2 - \beta^2}{2}\int_0^1 y^2(t)dt\right]\right)\,,$$

where $dy(t) = -\gamma y(t)dt + dw(t)$ with $y(0) = 0$.

On the basis of Theorem 4.1 it is an easy matter to compute $\phi_{S_1}(\theta)$. Let us first consider the case where $X(0) = Y(0) = \kappa$, a constant. Since $E(f(X)) = E(f(Y)d\mu_X(Y)/d\mu_Y)$, we obtain (Liptser and Shiryayev (1978, p.208) and Problem

1.2)

$$(4.8)\qquad \begin{aligned} E(e^{\theta S_1}) &= E\left[\exp\left\{\theta\int_0^1 X^2(t)dt\right\}\right] \\ &= E\left[\exp\left\{\theta\int_0^1 Y^2(t)dt\right\}\frac{d\mu_X}{d\mu_Y}(Y)\right] \\ &= E\left[\exp\left\{\left(\theta-\frac{\alpha^2-\beta^2}{2}\right)\int_0^1 Y^2(t)dt+(\beta-\alpha)\int_0^1 Y(t)dY(t)\right\}\right] \\ &= E\left[\exp\left\{\frac{\beta-\alpha}{2}(Y^2(1)-\kappa^2-1)\right\}\right] \\ &= \exp\left[\frac{\alpha}{2}+\frac{\kappa^2\theta\,\frac{\sinh\beta}{\beta}}{\cosh\beta+\alpha\,\frac{\sinh\beta}{\beta}}\right]\left[\cosh\beta+\alpha\,\frac{\sinh\beta}{\beta}\right]^{-\frac{1}{2}}, \end{aligned}$$

where $\beta=\sqrt{\alpha^2-2\theta}$ and we have used the fact that $Y(1)\sim \mathrm{N}(\kappa e^{-\beta},\,(1-e^{-2\beta})/(2\beta))$. Therefore we obtain

$$(4.9)\qquad \begin{aligned} \phi_{S_1}(\theta) &= E\left[\exp\left\{i\theta\int_0^1 X^2(t)dt\right\}\right] \\ &= \exp\left[\frac{\alpha}{2}+\frac{i\kappa^2\theta\,\frac{\sin\lambda}{\lambda}}{\cos\lambda+\alpha\,\frac{\sin\lambda}{\lambda}}\right]\left[\cos\lambda+\alpha\,\frac{\sin\lambda}{\lambda}\right]^{-\frac{1}{2}}, \end{aligned}$$

where $\lambda=\sqrt{2i\theta-\alpha^2}$.

Consider a special case where $\alpha=0$ so that

$$X(t)=\kappa+w(t)\Longleftrightarrow dX(t)=dw(t),\quad X(0)=\kappa\,.$$

Then we have, from (4.9),

$$E\left[\exp\left\{i\theta\int_0^1(\kappa+w(t))^2dt\right\}\right]=\exp\left\{i\kappa^2\theta\,\frac{\tan\sqrt{2i\theta}}{\sqrt{2i\theta}}\right\}\left(\cos\sqrt{2i\theta}\right)^{-\frac{1}{2}}.$$

A further special case of (4.9) with $\alpha=\kappa=0$ so that $X(t)=w(t)$ leads us to

$$(4.10)\qquad E\left[\exp\left\{i\theta\int_0^1 w^2(t)dt\right\}\right]=(\cos\sqrt{2i\theta})^{-\frac{1}{2}},$$

which was formally presented in Section 1 of Chapter 1.

As the second example we consider

$$S_2=\int_0^1\left\{X(t)-\int_0^1 X(s)ds\right\}^2dt=\int_0^1 X^2(t)dt-\left(\int_0^1 X(t)dt\right)^2,$$

where $\{X(t)\}$ is the O-U process defined in (4.2) with $X(0) = \kappa$. Proceeding in the same way as before we obtain (Problem 1.3)

$$\begin{aligned}
(4.11) \qquad E\left(e^{\theta S_2}\right) &= E\left[\exp\left\{\frac{\beta-\alpha}{2}(Y^2(1) - Y^2(0) - 1) - \theta\left(\int_0^1 Y(t)dt\right)^2\right\}\right] \\
&= \exp\left[\frac{1}{2}(\alpha-\beta)(\kappa^2+1)\right] \\
&\qquad \times E\left[\exp\left\{\frac{\beta-\alpha}{2}Y^2(1) - \theta\left(\int_0^1 Y(t)dt\right)^2\right\}\right] \\
&= \exp\left[\frac{\alpha}{2} + \frac{\alpha^2\kappa^2\theta}{g(\theta)}\left\{\frac{1}{\beta^2}\frac{\sinh\beta}{\beta} - \frac{2}{\beta^4}(\cosh\beta - 1)\right\}\right](g(\theta))^{-\frac{1}{2}},
\end{aligned}$$

where $\beta = \sqrt{\alpha^2 - 2\theta}$ and

$$g(\theta) = \frac{\alpha^3 - 2\theta}{\beta^2}\frac{\sinh\beta}{\beta} + \left(\frac{\alpha^2}{\beta^2} - \frac{4\alpha\theta}{\beta^4}\right)\cosh\beta + \frac{4\alpha\theta}{\beta^4}.$$

It can be checked that, when $\alpha = \kappa = 0$,

$$(4.12) \qquad E\left[\exp\left\{i\theta\int_0^1\left(w(t) - \int_0^1 w(s)ds\right)^2 dt\right\}\right] = \left(\frac{\sin\sqrt{2i\theta}}{\sqrt{2i\theta}}\right)^{-\frac{1}{2}},$$

which is the c.f. associated with the demeaned Brownian motion. In connection with (4.12) we can obtain (Problem 1.4) that

$$(4.13) \qquad E\left[\exp\left\{i\theta\int_0^1(w(t) - tw(1))^2dt\right\}\right] = \left(\frac{\sin\sqrt{2i\theta}}{\sqrt{2i\theta}}\right)^{-\frac{1}{2}},$$

which is the c.f. associated with the Brownian bridge and was formally presented in Section 2 of Chapter 1. The equivalence of the left sides of (4.12) and (4.13) has now been established.

In the above arguments we have assumed the initial value $X(0)$ to be a constant. Suppose now that $X(0) \sim \mathrm{N}(0, 1/(2\alpha))$ with $\alpha > 0$ so that $\{X(t)\}$ is stationary. Then Theorem 4.1 yields (Problem 1.5)

$$\begin{aligned}
(4.14) \qquad E\left(e^{\theta S_1}\right) &= E\left[\exp\left\{\frac{\beta-\alpha}{2}(Y^2(1) - Y^2(0) - 1)\right\}\right] \\
&= e^{\frac{\alpha}{2}}\left[\cosh\beta + \frac{\alpha^2-\theta}{\alpha}\frac{\sinh\beta}{\beta}\right]^{-\frac{1}{2}},
\end{aligned}$$

$$(4.15)\qquad E\left(e^{\theta S_2}\right) = E\left[\exp\left\{\frac{\beta-\alpha}{2}(Y^2(1)-Y^2(0)-1)-\theta\left(\int_0^1 Y(t)dt\right)^2\right\}\right]$$
$$= e^{\frac{\alpha}{2}}\left[\left(1+\frac{2\theta}{\beta^2}-\frac{2\alpha\theta}{\beta^4}\right)\cosh\beta + \left(\alpha+\frac{(\alpha-2)\theta}{\beta^2}\right)\frac{\sinh\beta}{\beta}+\frac{2\alpha\theta}{\beta^4}\right]^{-\frac{1}{2}},$$

where $\beta = \sqrt{\alpha^2 - 2\theta}$. A much simpler way of deriving (4.14) and (4.15) is possible since we have already obtained $E\left(e^{\theta S_1}\right)$ and $E\left(e^{\theta S_2}\right)$ when $X(0) = \kappa$. Replacing κ in (4.8) by $X(0)$ and noting that $E\left[\exp\left\{\theta X^2(0)\right\}\right] = (1-\theta/\alpha)^{-\frac{1}{2}}$, we have, in the present case,

$$E\left(e^{\theta S_1}\right) = E\left[E\left\{e^{\theta S_1}\middle| X(0)\right\}\right]$$
$$= e^{\frac{\alpha}{2}}\left[\left(1-\frac{\theta}{\alpha g(\theta)}\frac{\sinh\beta}{\beta}\right)g(\theta)\right]^{-\frac{1}{2}}$$
$$= e^{\frac{\alpha}{2}}\left[g(\theta)-\frac{\theta}{\alpha}\frac{\sinh\beta}{\beta}\right]^{-\frac{1}{2}},$$

where $g(\theta) = \cosh\beta + \alpha\sinh\beta/\beta$. This gives us (4.14). The derivation of (4.15) can be done similarly from (4.11).

The present approach can also be applied to obtain the c.f. corresponding to a random variable in ratio form. Let us consider

$$(4.16)\qquad R_1 = \frac{\int_0^1 X(t)dX(t)}{\int_0^1 X^2(t)dt} = \frac{U}{S_1},$$

where $\{X(t)\}$ is the O-U process defined in (4.2) with $X(0) = \kappa$.

The statistic R_1 may be regarded as the MLE of $-\alpha$ for the O-U process if the likelihood $l(-\alpha)$ for $\{X(t)\}$ is interpreted as

$$l(-\alpha) = \exp\left[-\alpha\int_0^1 X(t)dX(t) - \frac{\alpha^2}{2}\int_0^1 X^2(t)dt\right] = \frac{d\mu_X}{d\mu_w}(X).$$

Since $P(R_1 \le x) = P(xS_1 - U \ge 0)$, we are led to compute (Perron (1991a) and Problem 1.6)

$$(4.17)\qquad E\left[\exp\{\theta(xS_1-U)\}\right] = E\left[\exp\left\{\theta\left(x\int_0^1 X^2(t)dt - \int_0^1 X(t)dX(t)\right)\right\}\right]$$

$$
\begin{aligned}
&= E\left[\exp\left\{\frac{\beta-\alpha-\theta}{2}(Y^2(1)-\kappa^2-1)\right\}\right] \\
&= \exp\left[\frac{\alpha+\theta}{2}+\frac{\kappa^2\theta\left(\alpha+\frac{\theta}{2}+x\right)\frac{\sinh\beta}{\beta}}{\cosh\beta+(\alpha+\theta)\frac{\sinh\beta}{\beta}}\right] \\
&\quad\times\left[\cosh\beta+(\alpha+\theta)\frac{\sinh\beta}{\beta}\right]^{-\frac{1}{2}},
\end{aligned}
$$

where $\beta = \sqrt{\alpha^2 - 2\theta x}$. The joint m.g.f. of U and S_1 can be easily obtained from (4.17) (Problem 1.7).

Moments of R_1 can also be computed following the formula (1.39) as

$$
E(R_1^k) = \frac{1}{(k-1)!}\int_0^\infty \theta_2^{k-1}\left.\frac{\partial^k \psi(\theta_1,\,-\theta_2)}{\partial\theta_1^k}\right|_{\theta_1=0} d\theta_2\,,
$$

where $\psi(\theta_1,\,-\theta_2) = E[\exp\{\theta_1 U - \theta_2 S_1\}] = m(-\theta_1;\theta_2/\theta_1)$ with $m(\theta;x)$ being the m.g.f. of $xS_1 - U$ given in (4.17).

When $X(t) = w(t)$ so that $\alpha = \kappa = 0$, we have

$$
\begin{aligned}
E\left[\exp\{i\theta(xS_1-U)\}\right] &= E\left[\exp\left\{i\theta\left(x\int_0^1 w^2(t)dt - \int_0^1 w(t)dw(t)\right)\right\}\right] \\
&= e^{\frac{i\theta}{2}}\left[\cos\sqrt{2i\theta x} + i\theta\frac{\sin\sqrt{2i\theta x}}{\sqrt{2i\theta x}}\right]^{-\frac{1}{2}},
\end{aligned}
\tag{4.18}
$$

which was first obtained by White (1958) and was discussed in Section 3 of Chapter 1.

If $X(0) \sim \mathrm{N}(0, 1/(2\alpha))$ with $\alpha > 0$, we obtain (Problem 1.8), by the conditional argument described below (4.15),

$$
E\left[\exp\{\theta(xS_1-U)\}\right] = \exp\left(\frac{\alpha+\theta}{2}\right)\left[\cosh\beta + \frac{2\alpha^2-\theta^2-2\theta x}{2\alpha}\,\frac{\sinh\beta}{\beta}\right]^{-\frac{1}{2}}, \tag{4.19}
$$

where $\beta = \sqrt{\alpha^2 - 2\theta x}$.

We note in passing (Problem 1.9) that the statistic R_1 in (4.16) naturally appears in

$$
\mathcal{L}(T(\hat\rho - 1)) \longrightarrow \mathcal{L}(R_1)\,, \tag{4.20}
$$

where $\hat{\rho}$ is the LSE of ρ in the near integrated model :

$$\text{(4.21)} \qquad \begin{aligned} y_j &= \rho y_{j-1} + \varepsilon_j\,, \qquad (j = 1, \cdots, T)\,, \\ \rho &= 1 - \frac{\alpha}{T}\,, \quad \{\varepsilon_j\} \sim \text{i.i.d.}(0,1)\,, \end{aligned}$$

with $y_0 = \sqrt{T}\kappa$ or $y_0 = \sqrt{T}X(0)$. Here we can recognize the usefulness of the present approach since it is quite complicated to obtain results like (4.17) and (4.19) by the eigenvalue approach. More general models than (4.21) will be considered in Chapter 7, where we also discuss approximations to the finite sample distribution of $\hat{\rho}$.

As another example of ratio statistics we consider

$$\text{(4.22)} \qquad R_2 = \frac{\int_0^1 X_1(t)dw_2(t)}{\int_0^1 X_1^2(t)dt} = \frac{V}{S_1}\,,$$

where $\{X_1(t)\}$ is the O-U process defined by $dX_1(t) = -\alpha X_1(t)dt + dw_1(t)$ with $X_1(0) = \kappa$ and $\{w_1(t), w_2(t)\}$ is the two-dimensional standard Brownian motion. It is seen that, given $\{w_1(t)\}$ or S_1, R_2 is conditionally normal with the conditional mean 0 and the conditional variance S_1^{-1}. This fact may be expressed in a sophisticated way (Phillips (1989)) as

$$\text{(4.23)} \qquad \mathcal{L}(R_2) = \mathcal{L}\left(\int_{S_1>0} \mathrm{N}(0, S_1^{-1})dQ(S_1^{-1})\right)\,,$$

where Q is the probability measure associated with S_1^{-1}. We also have that, for any real x, $xS_1 - V$ is conditionally normal with

$$E\left[\,xS_1 - V \mid \{w_1(t)\}\right] = xS_1\,, \quad V[xS_1 - V \mid \{w_1(t)\}] = S_1\,.$$

Thus, using (4.9), we can easily derive

$$\text{(4.24)} \qquad \begin{aligned} E[\exp\{i\theta(xS_1 - V)\}] &= E\left[\exp\left\{i\left(\theta x + \frac{i\theta^2}{2}\right)S_1\right\}\right] \\ &= \exp\left[\frac{\alpha}{2} + \frac{i\kappa^2\theta\left(x + \frac{i\theta}{2}\right)\frac{\sin\lambda}{\lambda}}{\cos\lambda + \alpha\,\frac{\sin\lambda}{\lambda}}\right] \\ &\qquad \times \left[\cos\lambda + \alpha\,\frac{\sin\lambda}{\lambda}\right]^{-\frac{1}{2}}\,, \end{aligned}$$

where $\lambda = \sqrt{2i\theta x - \theta^2 - \alpha^2}$.

Similarly, when $X_1(0) \sim \mathrm{N}(0, 1/(2\alpha))$ with $\alpha > 0$, and $X_1(0)$ is independent of $\{w_2(t)\}$ and increments of $\{w_1(t)\}$, we obtain (Problem 1.10), using (4.14),

$$E[\exp\{i\theta(xS_1 - V)\}] = e^{\frac{\alpha}{2}} \left[\cos\lambda + \frac{\alpha^2 - i\theta\left(x + \frac{i\theta}{2}\right)}{\alpha} \frac{\sin\lambda}{\lambda} \right]^{-\frac{1}{2}}, \tag{4.25}$$

where $\lambda = \sqrt{2i\theta x - \theta^2 - \alpha^2}$.

The statistic R_2 in (4.22) naturally appears in (Problem 1.11)

$$\mathcal{L}(T(\hat{\beta} - \beta)) \longrightarrow \mathcal{L}(R_2), \tag{4.26}$$

where $\hat{\beta}$ is the LSE of β in the model :

$$\begin{aligned} y_{2j} &= \beta y_{1j} + \varepsilon_{2j}, \\ y_{1j} &= \left(1 - \frac{\alpha}{T}\right) y_{1,j-1} + \varepsilon_{1j}, \qquad (j = 1, \cdots, T), \end{aligned} \tag{4.27}$$

where $(\varepsilon_{1j}, \varepsilon_{2j})' \sim \text{i.i.d.}(0, I_2)$ and $y_{10} = \sqrt{T}\kappa$ or $y_{10} \sim \mathrm{N}(0, T/(2\alpha))$. In Section 5 of Chapter 1 we considered a simpler model with $\alpha = \kappa = 0$, which is a simplified version of the cointegrated system, and the c.f. corresponding to (4.24) was obtained in (1.83).

Problems

1.1 Establish the weak convergence result in (4.7).

1.2 Derive the m.g.f. in (4.8).

1.3 Derive the m.g.f. in (4.11) noting that

$$Y(t) = \kappa e^{-\beta t} + e^{-\beta t} \int_0^t e^{\beta s} dw(s),$$

$$\int_0^1 Y(t)dt = (1 - e^{-\beta})\frac{\kappa}{\beta} - \frac{1}{\beta} \int_0^1 \left(e^{-\beta(1-t)} - 1\right) dw(t).$$

1.4 Establish the result in (4.13).

1.5 Derive the m.g.f.'s in (4.14) and (4.15).

1.6 Derive the m.g.f. in (4.17).

1.7 Derive the joint m.g.f. of U and S_1 defined in (4.16).

1.8 Derive the m.g.f. in (4.19) by the conditional argument described below (4.15).

1.9 Establish the weak convergence result in (4.20).

1.10 Derive the m.g.f. in (4.25).

1.11 Establish the weak convergence result in (4.26).

4.2. Girsanov's theorem : case 2

In this section we extend Girsanov's theorem 4.1 to cover the case where $\{X(t)\}$ is the g-fold integrated Brownian motion ($g \geq 1$). Thus we consider

$$(4.28) \qquad F_g(t) = \int_0^t F_{g-1}(t)dt\,, \qquad F_0(t) = w(t)$$

and put

$$(4.29) \qquad dY_g(t) = \beta Y_g(t)dt + dF_g(t) = (\beta Y_g(t) + F_{g-1}(t))dt, \quad Y_g(0) = 0\,.$$

Our purpose here is to obtain the m.g.f. $m_g(\theta)$ of

$$(4.30) \qquad S(F_g) = \int_0^1 F_g^2(t)dt\,, \qquad (g \geq 1)\,.$$

If it holds that $m_g(\theta) = E[\exp\{\theta S(Y_g)d\mu_{F_g}(Y_g)/d\mu_{Y_g}\}]$, where μ_{F_g} and μ_{Y_g} are measures induced by $\{F_g(t)\}$ and $\{Y_g(t)\}$, respectively, then the computation of $m_g(\theta)$ may be feasible. For this purpose we establish the following theorem.

Theorem 4.2. *Let $\{F_g(t)\}$ and $\{Y_g(t)\}$ be defined by (4.28) and (4.29), respectively. Then probability measures μ_{F_g} and μ_{Y_g} are equivalent and*

$$(4.31) \qquad \frac{d\mu_{F_g}}{d\mu_{Y_g}}(y) = \exp\left[-\beta \int_0^1 \frac{d^g\, y(t)}{dt^g}\, d\left(\frac{d^g\, y(t)}{dt^g}\right) + \frac{\beta^2}{2}\int_0^1 \left(\frac{d^g\, y(t)}{dt^g}\right)^2 dt\right]\,,$$

where $y(0) = 0$ and $y \in C^{(g)}$ — the space of g-times continuously differentiable functions on [0,1].

Proof. It follows from (4.29) that

$$Y_g(t) = e^{\beta t} \int_0^t e^{-\beta s} F_{g-1}(s) ds\,, \tag{4.32}$$

which is g-times continuously differentiable so that it holds (Problem 2.1) that

$$Z(t) \equiv \frac{d^g\, Y_g(t)}{dt^g} = \beta \int_0^t Z(s) ds + w(t) \tag{4.33}$$

and thus

$$dZ(t) = \beta Z(t) dt + dw(t)\,, \qquad Z(0) = 0\,. \tag{4.34}$$

The measures μ_w and μ_Z are evidently equivalent by Theorem 4.1 and, by the same theorem, we have

$$\rho(x) \equiv \frac{d\mu_w}{d\mu_Z}(x) = \exp\left[-\beta \int_0^1 x(t) dx(t) + \frac{\beta^2}{2} \int_0^1 x^2(t) dt \right] \tag{4.35}$$

for $x \in C$ with $x(0) = 0$. Noting that

$$\frac{d^g\, F_g(t)}{dt^g} = w(t)\,, \qquad \frac{d^g\, Y_g(t)}{dt^g} = Z(t)\,,$$

we may put $F_g(t) = \Phi_g(w)(t)$ and $Y_g(t) = \Phi_g(Z)(t)$, where

$$\Phi_g(x)(t) = \int_0^t \Phi_{g-1}(x)(s) ds\,, \quad \Phi_0(x)(t) = x(t)\,, \quad x \in C\,. \tag{4.36}$$

Since $\mu_{F_g}(A) = P(\omega : F_g \in A) = P(\omega : w \in \Phi_g^{-1}(A))$ for $A \in \mathcal{B}(C^{(g)})$, we have $\mu_{F_g} = \mu_w \Phi_g^{-1}$. Similarly we have $\mu_{Y_g} = \mu_Z \Phi_g^{-1}$. Thus measures μ_{F_g} and μ_{Y_g} are equivalent and

$$\frac{d\mu_{F_g}}{d\mu_{Y_g}}(y) = \rho\left(\Phi_g^{-1}(y)\right) = \frac{d\mu_w}{d\mu_Z}\left(\Phi_g^{-1}(y)\right)\,, \quad y \in C^{(g)}$$

which establishes the theorem since $\Phi_g^{-1}(y)(t) = d^g\, y(t)/dt^g$ because of (4.36).

A heuristic derivation of the above theorem follows. Consider the discrete-time processes defined by

$$y_j = \left(1 + \frac{\beta}{T}\right) y_{j-1} + \frac{\varepsilon_j}{(1-L)^g}\,, \tag{4.37}$$

$$(4.38) \qquad z_j = (1-L)^g y_j = \left(1+\frac{\beta}{T}\right) z_{j-1} + \varepsilon_j\,, \quad (j=1,\cdots,T)\,,$$

where $\{\varepsilon_j\} \sim \mathrm{NID}(0,1)$ and $y_j = z_j = 0$ for $j \le 0$. Note that, when $\beta = 0$, $(1-L)^{g+1} y_j = \varepsilon_j$ and $(1-L)z_j = \varepsilon_j$. Let $l_T(0)$ and $l_T(\beta)$ be the likelihoods for $(y_1,\cdots,y_T)$ under $\beta = 0$ and $\beta \ne 0$, respectively. Then we can show (Problem 2.2) that

$$(4.39) \qquad \mathcal{L}\left(\left.\frac{l_T(0)}{l_T(\beta)}\right|_{\beta}\right) \longrightarrow \mathcal{L}\left(\exp\left[-\beta\int_0^1 Z(t)dZ(t) + \frac{\beta^2}{2}\int_0^1 Z^2(t)dt\right]\right)\,,$$

where $Z(t)$ is defined in (4.33).

We now consider $E[\exp\{\theta S(F_g)\}] = E[\exp\{\theta S(Y_g) d\mu_{F_g}(Y_g)/d\mu_{Y_g}\}]$, which involves the computation of

$$(4.40) \qquad \begin{aligned} Z(t) &= \frac{d^g\,Y_g(t)}{dt^g} = \beta\,\frac{d^{g-1}\,Y_g(t)}{dt^{g-1}} + w(t) \\ &= \beta^g Y_g(t) + \beta^{g-1} F_{g-1}(t) + \beta^{g-2} F_{g-2}(t) + \cdots + \beta F(t) + w(t)\,. \end{aligned}$$

The general case is evidently difficult to deal with. Let us restrict our attention to the case $g = 1$ and consider (Problem 2.3)

$$(4.41) \qquad \begin{aligned} m_1(\theta) &= E\left[\exp\left\{\theta\int_0^1 F_1^2(t)dt\right\}\right] \\ &= E\left[\exp\left\{\theta\int_0^1 Y_1^2(t)dt\right\}\frac{d\mu_{F_1}}{d\mu_{Y_1}}(Y_1)\right] \\ &= E\Bigg[\exp\Bigg\{\theta\int_0^1 Y_1^2(t)dt - \beta\int_0^1 \frac{dY_1(t)}{dt}\,d\left(\frac{dY_1(t)}{dt}\right) \\ &\qquad\qquad + \frac{\beta^2}{2}\int_0^1 \left(\frac{dY_1(t)}{dt}\right)^2 dt\Bigg\}\Bigg] \\ &= E\Bigg[\exp\Bigg\{\frac{\beta^2}{2}\int_0^1 w^2(t)dt - \frac{\beta}{2}w^2(1) \\ &\qquad\qquad -\beta^2 w(1) e^{\beta}\int_0^1 e^{-\beta t} w(t)dt + \frac{\beta}{2}\Bigg\}\Bigg]\,, \end{aligned}$$

where $\beta = (2\theta)^{\frac{1}{4}}$. We are in the same situation as was discussed in the last section. Define $dX(t) = \gamma X(t)dt + dw(t)$ with $X(0) = 0$ and apply Theorem 4.1 to obtain

(Problem 2.4)

$$(4.42)\qquad m_1(\theta) = E\left[\exp\left\{-\frac{\beta+\gamma}{2}X^2(1) - \beta^2 X(1)e^{\beta}\int_0^1 e^{-\beta t}X(t)dt + \frac{\beta+\gamma}{2}\right\}\right]$$
$$= \left[\frac{1}{2}\left\{1+\cos(2\theta)^{\frac{1}{4}}\ \cosh(2\theta)^{\frac{1}{4}}\right\}\right]^{-\frac{1}{2}},$$

where $\gamma = i\beta$.

The present approach can also be applied to derive the m.g.f.'s associated with ratio statistics. Let us first consider

$$(4.43)\qquad R_1 = \frac{\int_0^1 F_1(t)dF_1(t)}{\int_0^1 F_1^2(t)dt} = \frac{U}{S(F_1)}.$$

Then we obtain (Problem 2.5)

$$(4.44)\qquad E[\exp\{\theta(xS(F_1)-U)\}]$$
$$= E\left[\exp\left\{\frac{\beta^2}{2}\int_0^1 w^2(t)dt - \frac{\beta}{2}w^2(1) - \frac{\theta}{2}Y_1^2(1) - \beta^2 w(1)Y_1(1) + \frac{\beta}{2}\right\}\right]$$
$$= E\left[\exp\left\{-\frac{\beta+\gamma}{2}X^2(1) - \frac{\theta}{2}e^{2\beta}Z^2 - \beta^2 e^{\beta}X(1)Z + \frac{\beta+\gamma}{2}\right\}\right]$$
$$= \left[\frac{1}{2}(1+\cos\beta\cosh\beta) + \frac{\beta^2}{4x}\left(\cosh\beta\,\frac{\sin\beta}{\beta} - \cos\beta\frac{\sinh\beta}{\beta}\right)\right]^{-\frac{1}{2}},$$

where $\beta = (2\theta x)^{\frac{1}{4}}$, $\gamma = i\beta$ and

$$(4.45)\qquad Z = \int_0^1 e^{-\beta t}X(t)dt.$$

The statistic R_1 in (4.43) arises in (Problem 2.6)

$$(4.46)\qquad \mathcal{L}(T(\hat{\rho}-1)) \longrightarrow \mathcal{L}(R_1),$$

where $\hat{\rho}$ is the LSE of ρ (=1) in the model :

$$(4.47)\qquad y_j = \rho y_{j-1} + v_j, \qquad v_j = v_{j-1} + \varepsilon_j, \qquad (j = 1, \cdots, T)$$

with $v_0 = y_0 = 0$ and $\{\varepsilon_j\} \sim$ i.i.d.$(0,1)$.

We also consider

$$(4.48)\qquad R_2 = \frac{\int_0^1 F_1(t)dw_2(t)}{\int_0^1 F_1^2(t)dt} = \frac{V}{S(F_1)},$$

where $\{w_2(t)\}$ is the standard Brownian motion independent of $\{F_1(t)\}$. Using the conditional argument given in the last section we can easily obtain

$$
\begin{aligned}
(4.49) \qquad E[\exp\{\theta(xS(F_1)-V)\}] &= E\left[\exp\left\{\left(\theta x + \frac{\theta^2}{2}\right) S(F_1)\right\}\right] \\
&= m_1\left(\theta x + \frac{\theta^2}{2}\right),
\end{aligned}
$$

where $m_1(\theta)$ is defined in (4.42).

The statistic R_2 in (4.48) arises in (Problem 2.7)

$$
(4.50) \qquad \mathcal{L}(T^2(\hat{\beta}-\beta)) \longrightarrow \mathcal{L}(R_2),
$$

where $\hat{\beta}$ is the LSE of β in the second-order cointegrated model :

$$
(4.51) \qquad
\begin{aligned}
& y_{2j} = \beta y_{1j} + \varepsilon_{2j}, \\
& (1-L)^2 y_{1j} = \varepsilon_{1j}, \qquad (j = 1, \cdots, T),
\end{aligned}
$$

with $(\varepsilon_{1j}, \varepsilon_{2j})' \sim$ i.i.d.$(0, I_2)$ and $y_{1,-1} = y_{10} = 0$.

The computation of the m.g.f.'s or c.f.'s for higher order integrated processes is much involved because of the complicated expression for $d^g Y_g(t)/dt^g$ given in (4.40). Even the case $g = 2$ turns out to be hard to deal with. In the next chapter we present another approach which makes the computation feasible by making use of computerizd algebra.

Problems

2.1 Prove that the process $\{Y_g(t)\}$ defined in (4.32) satisfies the relation in (4.33).

2.2 Establish the weak convergence result in (4.39).

2.3 Derive the expressions in (4.41).

2.4 Obtain the m.g.f. $m_1(\theta)$ as in (4.42).

2.5 Derive the m.g.f. in (4.44).

2.6 Establish the weak convergence result in (4.46).

2.7 Establish the weak convergence result in (4.50).

4.3. Girsanov's theorem : case 3

In this section we deal with the q-dimensional standard Brownian motion $\{w(t)\}$ and consider

$$dX(t) = AX(t)dt + dw(t)\,, \tag{4.52}$$

$$dY(t) = BY(t)dt + dw(t)\,, \quad X(0) = Y(0) = 0\,, \tag{4.53}$$

where A and B are $q \times q$ constant matrices. The processes $\{X(t)\}$ and $\{Y(t)\}$ are q-dimensional O-U processes, for which it is known (Arnold (1974, p.129)) that

$$X(t) = e^{At}\int_0^t e^{-As}dw(s)\,, \quad Y(t) = e^{Bt}\int_0^t e^{-Bs}dw(s)\,. \tag{4.54}$$

Note that e^{At} is a matrix-valued function of t defined by

$$e^{At} = \sum_{n=0}^{\infty}\frac{t^n}{n!}A^n\,. \tag{4.55}$$

Later we shall also introduce matrix-valued functions $\cosh At = (e^{At} + e^{-At})/2$, $\sinh At = (e^{At} - e^{-At})/2$, $\tanh At = (\cosh At)^{-1}\sinh At$ and so on.

Girsanov's theorem still applies to the above situation and is stated as follows (Liptser and Shiryayev (1977, p.279)).

Theorem 4.3. *Let μ_X and μ_Y be probability measures induced by $\{X(t)\}$ and $\{Y(t)\}$, respectively. Then μ_X and μ_Y are equivalent and*

$$\begin{aligned}\frac{d\mu_X}{d\mu_Y}(x) \;&=\; \exp\left[\int_0^1 x'(t)(A-B)'dx(t)\right.\\ &\qquad\left. -\frac{1}{2}\int_0^1 x'(t)(A-B)'(A+B)x(t)dt\right]\,,\end{aligned} \tag{4.56}$$

where $x \in C^q$ with $x(0) = 0$.

As an application of this theorem we derive the m.g.f. of

$$S_1 = \int_0^1 w'(t)Hw(t)dt\,, \tag{4.57}$$

where H is a $q \times q$ symmetric matrix. Of course it is easier in this case to use the result in (4.10) for the scalar case and the independence property of components of $\{w(t)\}$, which yields (Problem 3.1)

$$E(e^{\theta S_1}) = \prod_{j=1}^{q} (\cos\sqrt{2\lambda_j \theta})^{-\frac{1}{2}}\,, \tag{4.58}$$

where λ_j's are the eigenvalues of H. Nonetheless we consider

$$\begin{aligned} E(e^{\theta S_1}) &= E\left[\exp\left\{\theta \int_0^1 X'(t)HX(t)dt\right\} \frac{d\mu_w}{d\mu_X}(X)\right] \\ &= E\left[\exp\left\{\int_0^1 X'(t)\left(\theta H + \frac{1}{2}A'A\right)X(t)dt \right.\right. \\ &\qquad \left.\left. - \int_0^1 X'(t)A'dX(t)\right\}\right]. \end{aligned}$$

Putting $A^2 = -2\theta H$ with A being symmetric and using the matrix version of Ito's theorem (Arnold (1974, p.143)) :

$$d(X(t)X'(t)) = X(t)dX'(t) + dX(t)X'(t) + I_q dt\,,$$

we obtain (Problem 3.2), if A is symmetric,

$$\begin{aligned} \int_0^1 X'(t)A'dX(t) &= \int_0^1 X'(t)A\,dX(t) \\ &= \frac{1}{2}[X'(1)AX(1) - \mathrm{tr}\,(A)]\,, \end{aligned} \tag{4.59}$$

from which we derive (4.58) (Problem 3.3).

It is an immediate consequence of the result in (4.58) to obtain, for example, the m.g.f. of

$$S_2 = \int_0^1 w_1(t)w_2(t)dt = \int_0^1 w'(t)Hw(t)dt\,, \tag{4.60}$$

where H is the 2×2 matrix with $H_{11} = H_{22} = 0$ and $H_{12} = H_{21} = \frac{1}{2}$. We have (Problem 3.4)

$$E(e^{\theta S_2}) = (\cos\sqrt{\theta}\cosh\sqrt{\theta})^{-\frac{1}{2}}\,. \tag{4.61}$$

The statistic S_2 in (4.60) was discussed in Section 4 of Chapter 1.

Another important and interesting statistic takes the following form:

$$S_3 = \int_0^1 w'(t) G dw(t)\,, \tag{4.62}$$

where G is any constant matrix. If G is symmetric, then we can use the relation in (4.59) and readily obtain the distribution of S_3 (Problem 3.5).

Two cases of G being not symmetric were also discussed in Section 4 of Chapter 1. One was

$$S_4 = \int_0^1 w_1(t) dw_2(t) = \int_0^1 w'(t) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} dw(t) \tag{4.63}$$

and it can be shown by the conditional argument (Problem 3.6) that $E(e^{\theta S_4}) = (\cos\theta)^{-\frac{1}{2}}$. The other was Lévy's stochastic area defined by

$$\begin{aligned} S_5 &= \frac{1}{2}\int_0^1 [w_1(t)dw_2(t) - w_2(t)dw_1(t)] \\ &= \int_0^1 w'(t) \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} dw(t)\,. \end{aligned} \tag{4.64}$$

It holds (Problem 3.7) that $E\,[\,S_5\,|\,\{w_1(t)\}\,] = 0$ and

$$V[S_5\,|\,\{w_1(t)\}] = \int_0^1 \left(w_1(t) - \frac{1}{2}w_1(1)\right)^2 dt\,. \tag{4.65}$$

We can now obtain (Problem 3.8) $E(e^{\theta S_5}) = \left(\cos\frac{\theta}{2}\right)^{-1}$.

Problems

3.1 Derive the m.g.f. of S_1 in (4.57) using the result in Section 1 and the independence property of components of $\{w(t)\}$.

3.2 Establish the relation in (4.59).

3.3 Derive the m.g.f. of S_1 in (4.57) using the relation in (4.59).

3.4 Obtain the m.g.f. of S_2 as in (4.61).

3.5 Derive the m.g.f. of S_3 in (4.62) when G is symmetric.

3.6 Derive the m.g.f. of S_4 in (4.63).

3.7 Compute the conditional variance of S_5 in (4.65) given $\{w_1(t)\}$.

3.8 Derive the m.g.f. of S_5 in (4.64) using (4.65) and the relations described in Problem 1.3.

4.4. The Cameron - Martin formula

Here we concentrate on quadratic functionals of the q-dimensional standard Brownian motion $\{w(t)\}$ and present a formula for computing the m.g.f.'s especially designed for such functionals. The following result, known as the Cameron-Martin formula, is established in Liptser and Shiryayev (1977, p.280).

Theorem 4.4. *Let $H(t)$ be a $q \times q$ symmetric nonnegative definite matrix whose elements $H_{jk}(t)$ are continuous and satisfy the condition*

$$\int_0^1 \sum_{j,k=1}^{q} |H_{jk}(t)| dt < \infty \,.$$

Then it holds that

$$E\left[\exp\left\{-\int_0^1 w'(t)H(t)w(t)dt\right\}\right] = \exp\left[\frac{1}{2}\int_0^1 tr\,(G(t))dt\right] , \tag{4.66}$$

where $G(t)$ is a $q \times q$ symmetric nonpositive definite matrix, being a unique solution of the matrix-valued Riccati differential equation :

$$\frac{dG(t)}{dt} = 2H(t) - G^2(t)\,, \qquad G(1) = 0\,. \tag{4.67}$$

It is difficult, in general, to solve the matrix equation in (4.67). Suppose that $H(t)$ is a constant matrix so that we put $H(t) = H$. Then it is known (Bellman (1970, p.323)) that the solution is given by

$$G(t) = D \tanh D(t-1) = D(\cosh D(t-1))^{-1} \sinh D(t-1)\,, \tag{4.68}$$

where D is a positive square root of $2H$, that is $D = (2H)^{\frac{1}{2}}$. Using the facts that

$$\frac{d \log(\cosh D(t-1))}{dt} = D \tanh D(t-1)\,, \tag{4.69}$$

$$\exp\left[\frac{1}{2}\mathrm{tr}\left\{\log(\cosh D)\right\}\right] = \prod_{j=1}^{q} \sqrt{\cosh \delta_j}\,, \tag{4.70}$$

where δ_j's are the eigenvalues of D, we can show that

$$E\left[\exp\left\{\theta \int_0^1 w'(t)Hw(t)dt\right\}\right] = \prod_{j=1}^{q} (\cos \sqrt{2\lambda_j \theta})^{-\frac{1}{2}}\,, \tag{4.71}$$

where λ_j's are the eigenvalues of H. This result was already obtained in the last section. Evidently the assumption of positive definiteness of H is not necessary.

For our purpose the usefulness of the Cameron-Martin formula crucially depends on the solvability of the matrix equation (4.67). Even for the scalar case it cannot be solved explicitly, in general. Consider, for example

$$S = \int_0^1 t^m w^2(t)dt\,. \tag{4.72}$$

Then Theorem 4.4 leads us to

$$E(e^{\theta S}) = \exp\left\{\frac{1}{2}\int_0^1 g(t)dt\right\}\,,$$

where $g(t)$ is the solution to

$$\frac{dg(t)}{dt} = -2\theta t^m - g^2(t)\,, \quad g(1) = 0\,.$$

The explicit solution may be obtained by quadrature for some m's after tedious efforts; then it must be integrated.

In the next chapter we consider another approach and obtain the m.g.f. of S in (4.72) for any m (> -1).

4.5. Advantages and disadvantages of the present approach

The success of the stochastic process approach crucially depends on the computability of the expectation of a functional of the Brownian motion, where the expectation is taken with respect to the transformed measure given by Girsanov's theorem. As we have seen, the present approach is quite successful in dealing with quadratic functionals of the O-U process. In fact it will be seen in the next chapter that the present approach is more suitable for the analysis of the O-U process than the approach introduced there.

In practice, however, we need to deal with other classes of functionals of the Brownian motion. As an example let us consider the process

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dw(t)\,, \tag{4.73}$$

which was introduced in Section 7 of Chapter 2 as the Ito stochastic differential equation. The existence of a unique solution to (4.73) was also discussed there. In connection with this process we consider an auxiliary process

$$dY(t) = m(Y(t),t)dt + \sigma(Y(t),t)dw(t)\,, \quad Y(0) = X(0)\,. \tag{4.74}$$

Then the two measures μ_X and μ_Y are equivalent under some suitable conditions and the Radon-Nikodym derivative is given in Liptser and Shiryayev (1977, p.277) by

$$\begin{aligned} \frac{d\mu_X}{d\mu_Y}(x) \;&=\; \exp\left[\int_0^1 \frac{\mu(x(t),t) - m(x(t),t)}{\sigma^2(x(t),t)}\,dx(t)\right. \\ &\qquad \left. - \frac{1}{2}\int_0^1 \frac{\mu^2(x(t),t) - m^2(x(t),t)}{\sigma^2(x(t),t)}\,dt\right], \end{aligned} \tag{4.75}$$

where $x \in C$ with $x(0) = X(0)$. Then it might be thought to be possible to compute $E(f(X)) = E(f(Y)d\mu_X(Y)/d\mu_Y)$ for a functional $f(X)$, but the computation turns out to be difficult.

To see the difficulty let us consider $dX(t) = t^k dw(t)$ with $X(0) = 0$, which is a special case of (4.73) with $\mu = 0$ and $\sigma = t^k$. Then, using (4.75), we shall have, for example,

$$E\left[\exp\left(\theta\int_0^1 X^2(t)dt\right)\right] = E\left[\exp\left\{-\sqrt{-2\theta}\int_0^1 \frac{Y(t)}{t^k}dY(t)\right\}\right], \tag{4.76}$$

where we have put $m(Y(t),t) = \sqrt{-2\theta}\,t^k Y(t)$. The stochastic process $\{Y(t)\}$ in the present case has the solution

$$Y(t) = \exp\left(\frac{\sqrt{-2\theta}\,t^{k+1}}{k+1}\right)\int_0^t \exp\left(-\frac{\sqrt{-2\theta}s^{k+1}}{k+1}\right)s^k dw(s)\,,$$

but this does not help in computing the right side of (4.76) except for $k = 0$.

In the next chapter we consider another approach which overcomes the above difficulty. Since

$$dX(t) = t^k dw(t)\,, \quad X(0) = 0 \Longleftrightarrow X(t) = \int_0^t s^k dw(s)\,,$$

the Riemann integral in (4.76) can be rewritten as

$$\begin{aligned} \int_0^1 X^2(t)dt \;&=\; \int_0^1\left\{\int_0^t\int_0^t u^k v^k dw(u)dw(v)\right\}dt \\ &=\; \int_0^1\int_0^1 [1 - \max(s,t)]\, s^k t^k dw(s)dw(t)\,. \end{aligned}$$

This last expression enables us to obtain the c.f. by the approach presented in the next chapter.

We have also noted in Section 4 that the stochastic process approach is not suitable for obtaining the c.f. of the following form :

$$
\begin{aligned}
S &= \int_0^1 t^{2k} w^2(t)dt \\
&= \int_0^1 \int_0^1 \frac{1}{2k+1} \left[1 - (\max(s,t))^{2k+1}\right] dw(s)dw(t)\,.
\end{aligned}
$$

This last expression will again make the derivation of the c.f. possible.