Time Series Analysis

Nonstationary and Noninvertible Distribution Theory

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Chapter 5 The Fredholm Approach

We present another method for computing the c.f.'s of quadratic plus linear or bilinear functionals of the Brownian motion, where functionals are expressed by the Riemann-Stieltjes integral. The method requires some knowledge from the theory of integral equations of Fredholm type, among which are the Fredholm determinant and the resolvent. We give an introductory discussion on these together with various examples of how to derive them. We then indicate by some theorems and examples how to relate the c.f. with the Fredholm determinant and the resolvent. It turns out that the present approach enables us to deal with a wider class of functionals than the stochastic process and eigenvalue approaches.

5.1. Motivating examples

In the first four sections of this chapter we deal with a statistic expressed in the Riemann-Stieltjes integral :

$$(5.1) \qquad S = \int_0^1 \int_0^1 K(s,t)dw(s)dw(t)\,,$$

where $K(s,t)(\not\equiv 0)$ is a function with some conditions imposed later, while $\{w(t)\}$ is the one-dimensional standard Brownian motion. The statistic S covers a wide class of quadratic functionals of the Brownian motion. As an example let $X(t)$ be the O-U process defined by $dX(t) = -\beta X(t)dt + dw(t)$ with $X(0) = 0$. Then we have (Problem 1.1)

$$(5.2) \qquad \int_0^1 X^2(t)dt = \int_0^1 \int_0^1 \frac{e^{-\beta|s-t|} - e^{-\beta(2-s-t)}}{2\beta}\, dw(s)dw(t)\,.$$

If $X(0)$ is a nonzero constant or a random variable, the integral on the right side of (5.2) contains a linear or bilinear functional of the Brownian motion as well. The quadratic plus linear or bilinear forms will be dealt with in Section 5.

As is seen above, the integral of the square of the O-U process with the initial value equal to 0 can always be expressed as in (5.1), and we have indicated in the last chapter that the c.f. of (5.2) can be easily obtained by the stochastic process approach. Thus, as far as the O-U process is concerned, there is no advantage of using the double integral expression as on the right side of (5.2). We, however, need to consider the other processes which do not fall into the O-U process. As an example let us consider

$$(5.3) \qquad S_1 = \int_0^1 (g(t)w(t))^2 dt = \int_0^1 \int_0^1 K(s,t)dw(s)dw(t)\,,$$

where $g(t)$ is a nonstochastic, continuous function and

$$K(s,t) = \int_{\max(s,t)}^1 g^2(u)du\,.$$

Here $Y(t) = g(t)w(t)$ is not the O-U process, and it is difficult to obtain the c.f. of S_1 by the stochastic process approach, as was discussed in Section 5 of Chapter 4.

Some other examples which motivate the double integral expression follow. In Section 3 of Chapter 4 we dealt with Lévy's stochastic area defined by

$$S_2 = \frac{1}{2}\int_0^1 [w_1(t)dw_2(t) - w_2(t)dw_1(t)]\,, \tag{5.4}$$

where $w(t) = (w_1(t), w_2(t))'$ is the two-dimensional standard Brownian motion. The expression on the right side of (5.4) is far from the double integral expression, but we have seen that, given $\{w_1(t)\}$, S_2 is conditionally normal with the mean 0 and the variance given by the right side of (4.65), which can be rewritten (Problem 1.2) as

$$V\left[S_2 \,|\, \{w_1(t)\}\right] = \int_0^1\int_0^1 \frac{1}{4}\left[1 - 2\,|s-t|\,\right] dw_1(s)dw_1(t)\,. \tag{5.5}$$

Then we can proceed to the computation of

$$E(e^{i\theta S_2}) = E\left[\exp\left\{-\frac{\theta^2}{2}\int_0^1\int_0^1 \frac{1}{4}\left[1 - 2\,|s-t|\,\right] dw_1(s)dw_1(t)\right\}\right]\,.$$

Returning to the one-dimensional standard Brownian motion $\{w(t)\}$, let us consider

$$S_3 = \int_0^1 g(t)w(t)dw(t)\,. \tag{5.6}$$

The conditional argument is not applicable here. Assuming $g(t)$ to be differentiable we use Ito's theorem to obtain

$$d(g(t)w^2(t)) = 2g(t)w(t)dw(t) + (g'(t)w^2(t) + g(t))dt$$

so that we have (Problem 1.3)

$$S_3 = \frac{1}{2}\int_0^1\int_0^1 g(\max(s,t))dw(s)dw(t) - \frac{1}{2}\int_0^1 g(t)dt\,. \tag{5.7}$$

We also consider the g-fold integrated Brownian motion $\{F_g(t)\}$ and put

$$\begin{aligned} S_4 &= \int_0^1 F_g^2(t)dt \\ &= \int_0^1\int_0^1 \left[\int_{\max(s,t)}^1 \frac{((u-s)(u-t))^g}{(g!)^2}\,du\right] dw(s)dw(t)\,. \end{aligned} \tag{5.8}$$

In Section 2 of Chapter 4 we have obtained the c.f. of S_4 for $g = 1$ by the stochastic process approach. For $g \geq 2$, however, that approach is much involved, as was mentioned there. We shall obtain the c.f. of S_4 for $g = 2$ using the last expression in (5.8).

The above examples deal directly with the Riemann-Stieltjes integrals. In some cases these integrals naturally emerge from quadratic forms under finite samples. Here we just take up one example. Let us consider

$$S_{T5} = \frac{1}{T^2}\varepsilon'\Sigma_T\,\varepsilon\,, \tag{5.9}$$

where $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_T)'$ with $\{\varepsilon_j\} \sim$ i.i.d.$(0,1)$, while Σ_T is the $T \times T$ matrix with the (j,k)-th element $\Sigma_T(j,k)$ being $\min(j,k) - (jk/T)$. Here it holds that, for $K(s,t) = \min(s,t) - st$,

$$\frac{1}{T}\Sigma_T(j,k) = K\left(\frac{j}{T}, \frac{k}{T}\right) \qquad \text{for all} \quad j,\ k\,.$$

Although we leave a rigorous treatment to Section 6, we can show that

$$\begin{aligned} \mathcal{L}(S_{T5}) &= \mathcal{L}\left(\frac{1}{T}\sum_{j,k=1}^{T} K\left(\frac{j}{T}, \frac{k}{T}\right)\varepsilon_j\varepsilon_k\right) \\ &\to \mathcal{L}\left(\int_0^1\int_0^1 [\min(s,t) - st\,]\, dw(s)dw(t)\right)\,. \end{aligned}$$

In the present case the limiting random variable may be expressed by the Riemann integral (Problem 1.4), but, given (5.9), the present expression seems more natural.

Thus we strongly feel the need of the study of how to compute the c.f. of the statistic that takes the form in (5.1). For this purpose we introduce the approach which we call the *Fredholm approach*, named after the Swedish mathematician E. I. Fredholm (1866-1927). To develop this approach we require some knowledge from the theory of *integral equations of Fredholm type*, which we now describe briefly following Courant and Hilbert (1953), Hochstadt (1973) and Whittaker and Watson (1958).

Problems

1.1 Establish the relation in (5.2).

1.2 Derive the expression in (5.5).

1.3 Establish the relation in (5.7).

1.4 Obtain the limiting form of $\mathcal{L}(S_{T5})$ in (5.9) by the FCLT discussed in Chapter 3.

5.2. The Fredholm theory : the homogeneous case

Let us consider the integral equation for λ and $f(t)$:

$$f(t) = \lambda \int_0^1 K(s,t) f(s) ds \,, \tag{5.10}$$

where $K(s,t)$ is a known, continuous function on $[0,1] \times [0,1]$. A value λ(possibly complex) for which this integral equation possesses a nonvanishing continuous solution is called an *eigenvalue* of the *kernel* $K(s,t)$; the corresponding solution $f(t)$ is called an *eigenfunction* for the eigenvalue λ. The maximum number l of linearly independent solutions is called the *multiplicity* of λ.

It is usually difficult to solve for λ and $f(t)$ analytically. For our purpose, however, it is not necessary. We have only to obtain the *Fredholm determinant*, the definition of which is given below.

The integral equation (5.10) may be approximated by the algebraic system

$$f\left(\frac{j}{T}\right) = \frac{\lambda}{T} \sum_{k=1}^{T} K\left(\frac{k}{T}, \frac{j}{T}\right) f\left(\frac{k}{T}\right) \qquad (j = 1, \cdots, T) \,,$$

or, in matrix notation,

$$f_T = \frac{\lambda}{T} K_T f_T \,, \tag{5.11}$$

where $f_T = (\!(f(j/T))\!)$ is a $T \times 1$ vector and $K_T = (\!(K(j/T, k/T))\!)$ is a $T \times T$ matrix. As in the theory of matrices we can consider the characteristic equation that determines λ. In the present case, however, we study the asymptotic behavior, as $T \to \infty$, of

$$D_T(\lambda) = |I_T - \frac{\lambda}{T} K_T| \,. \tag{5.12}$$

Clearly, $D_T(\lambda)$ is a polynomial of degree T in λ. Thus we may put

$$D_T(\lambda) = \sum_{l=0}^{T} \frac{a_l(T)}{l!} \lambda^l \,, \qquad a_l(T) = \frac{d^l}{d\lambda^l} D_T(\lambda) \bigg|_{\lambda=0} \,. \tag{5.13}$$

Clearly $a_0(T) = 1$ and we have (Problem 2.1)

$$(5.14)\qquad a_1(T) = -\frac{1}{T}\sum_{j=1}^{T} K\left(\frac{j}{T}, \frac{j}{T}\right) \to -\int_0^1 K(t,t)dt\,,$$

$$(5.15)\qquad a_2(T) = \frac{1}{T^2}\sum_{j,k=1}^{T} \begin{vmatrix} K\left(\frac{j}{T}, \frac{j}{T}\right) & K\left(\frac{j}{T}, \frac{k}{T}\right) \\ K\left(\frac{k}{T}, \frac{j}{T}\right) & K\left(\frac{k}{T}, \frac{k}{T}\right) \end{vmatrix}$$
$$\to \int_0^1\int_0^1 \begin{vmatrix} K(s,s) & K(s,t) \\ K(t,s) & K(t,t) \end{vmatrix} dsdt\,.$$

More generally it can be shown (Hochstadt (1973, p.237)) that

$$a_l(T) = \frac{(-1)^l}{T^l}\sum_{j_1,\cdots,j_l} K\begin{pmatrix} \frac{j_1}{T} & \cdot & \cdot & \cdot & \frac{j_l}{T} \\ \frac{j_1}{T} & \cdot & \cdot & \cdot & \frac{j_l}{T} \end{pmatrix}$$
$$\to (-1)^l \int_0^1 \cdots \int_0^1 K\begin{pmatrix} t_1 & \cdot & \cdot & \cdot & t_l \\ t_1 & \cdot & \cdot & \cdot & t_l \end{pmatrix} dt_1\cdots dt_l\,,$$

where

$$(5.16)\qquad K\begin{pmatrix} t_1 & \cdot & \cdot & \cdot & t_l \\ t_1 & \cdot & \cdot & \cdot & t_l \end{pmatrix} = \begin{vmatrix} K(t_1,t_1) & \cdot & \cdot & \cdot & K(t_1,t_l) \\ \vdots & & & & \vdots \\ K(t_l,t_1) & \cdot & \cdot & \cdot & K(t_l,t_l) \end{vmatrix}.$$

Then it can also be derived from (5.13) that

$$(5.17)\qquad D(\lambda) = \lim_{T\to\infty} D_T(\lambda)$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n\lambda^n}{n!}\int_0^1 \cdots \int_0^1 K\begin{pmatrix} t_1 & \cdot & \cdot & \cdot & t_n \\ t_1 & \cdot & \cdot & \cdot & t_n \end{pmatrix} dt_1\cdots dt_n\,.$$

The function $D(\lambda)$ is called the *Fredholm determinant* (FD) of the kernel $K(s,t)$. It holds that the series in (5.17) converges for all λ, that is, $D(\lambda)$ is an *entire* or *integral* function with $D(0) = 1$. In fact it holds (Hochstadt (1973, p.239)) that

$$|D(\lambda)| \le \sum_{n=0}^{\infty} \frac{(|\lambda| M\sqrt{n})^n}{n!}\,,$$

where $M = \max|K(s,t)|$ and the series on the right side can be seen to converge for all λ by the ratio test.

Some properties of entire functions follow. Let $h(z)$ be an entire function of z which may be complex with $h(0) = 1$; and let the zeros of $h(z)$ be at $a_1, a_2, \cdots$, where $\lim_{n\to\infty} |a_n| = \infty$ and the zero at a_n be of order m_n. Then $h(z)$ can be expanded (Whittaker and Watson (1958, p.139)) as

$$(5.18) \qquad h(z) = e^{G(z)} \prod_{n=1}^{\infty} \left[\left\{ \left(1 - \frac{z}{a_n}\right) e^{g_n(z)} \right\}^{m_n} \right],$$

where $G(z)$ is some entire function such that $G(0) = 0$, while

$$g_n(z) = \frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{k_n}\left(\frac{z}{a_n}\right)^{k_n}$$

with k_n being the smallest integer such that

$$\left| m_n \left(\frac{N}{a_n}\right)^{k_n+1} \right| < b_n, \qquad \sum_{n=1}^{\infty} b_n < \infty,$$

for a constant N. We put $g_n(z) = 0$ if $k_n = 0$. If it is possible to choose all the k_n equal to each other, then $k = k_n$ is called the *genus* associated with the infinite product.

In particular, if $D(\lambda)$ is the FD of a continuous kernel $K(s,t)$ with an infinite number of eigenvalues $\{\lambda_n\}$, the infinite product takes the form (Hochstadt (1973, p.249))

$$(5.19) \qquad D(\lambda) = \exp\left\{-\lambda \int_0^1 K(t,t)dt\right\} \prod_{n=1}^{\infty} \left[\left\{ \left(1 - \frac{\lambda}{\lambda_n}\right) \exp\left(\frac{\lambda}{\lambda_n}\right) \right\}^{m_n} \right],$$

where m_n is the order of the zero of $D(\lambda)$ at λ_n. Thus $D(\lambda)$ in the present case is an entire functin of genus unity. Note that m_n is not necessarily equal to the multiplicity of λ_n. A much simpler representation for $D(\lambda)$ with genus zero and m_n equal to the multiplicity of λ_n will be obtained later by imposing some conditions on $K(s,t)$.

The following theorem holds because of (5.19) concerning the relationship between the zeros of $D(\lambda)$ and the eigenvalues of $K(s,t)$.

Theorem 5.1. *Every zero of $D(\lambda)$ is an eigenvalue of K, and in turn every eigenvalue of K is a zero of $D(\lambda)$.*

Note that zero is never an eigenvalue since $D(0) = 1 \neq 0$. It sometimes happens that $D(\lambda)$ never becomes zero so that there exists no eigenvalue. It is known, however, that, if $K(s,t)$ is symmetric as well as continuous, then there exists at least one eigenvalue as far as $K(s,t)$ is not identically equal to zero.

In subsequent discussions we assume $K(s,t)$ to be continuous and symmetric on $[0,1] \times [0,1]$. Then every eigenvalue is real. If there are an infinite number of eigenvalues, $K(s,t)$ is said to be *nondegenerate*; otherwise it is *degenerate*. When $K(s,t)$ is nondegenerate, $\lambda = \infty$ is the only accumulation point of zeros. If all the eigenvalues of K have the same sign, then $K(s,t)$ is said to be *definite*. Alternatively, $K(s,t)$ is positive (negative) definite if $\int_0^1 \int_0^1 K(s,t)g(s)g(t)dsdt$ is nonnegative (nonpositive) for any continuous function $g(t)$ on [0,1]. If all but a finite number of eigenvalues have the same sign, $K(s,t)$ is said to be *nearly definite*. A necessary and sufficient condition for $K(s,t)$ to be nearly definite is that it can be expressed as the sum of a definite kernel and degenerate kernels.

Some examples of $K(s,t)$ follow. Consider the following functions:

$$K_1(s,t) = 1 - \max(s,t)\,, \quad K_2(s,t) = \min(s,t) - st\,, \quad K_3(s,t) = g(s)g(t)\,, \tag{5.20}$$

where $g(t)$ is continuous. Then it can be shown (Problem 2.2) that these are all positive definite. It will be recognized later that K_3 is degenerate, while K_1 and K_2 are nondegenerate.

Suppose that $K(s,t)$ is nearly definite, which we shall also assume in subsequent discussions. Then the following theorem called *Mercer's theorem* holds (Hochstadt (1973, p.91)).

Theorem 5.2. *Let $K(s,t)$ be continuous, symmetric and nearly definite on $[0,1] \times [0,1]$. Then*

$$K(s,t) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} f_n(s) f_n(t)\,, \tag{5.21}$$

where $\{\lambda_n\}$ is a sequence of eigenvalues repeated as many times as their multiplicities, while $\{f_n(t)\}$ is an orthonormal sequence of eigenfunctions corresponding to eigenvalues λ_n and the series on the right side converges absolutely and uniformly to $K(s,t)$.

Mercer's theorem tells us (Problem 2.3) that

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \int_0^1 K(t,t)dt\,, \tag{5.22}$$

where λ_n is repeated as many times as its multiplicity. Mercer's theorem will also be effectively used for deriving the c.f. of a quadratic functional of the Brownian motion.

Recalling the infinite product representation for $D(\lambda)$ in (5.19) together with (5.22) leads us to the following theorem (Hochstadt (1973, p.251)).

Theorem 5.3. *Suppose that $K(s,t)$ is continuous, symmetric and nearly definite on $[0,1]\times[0,1]$. Then the FD of K can be expanded as*

$$D(\lambda) = \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right)^{l_n}\,, \tag{5.23}$$

where λ_n is the eigenvalue of K with $\lambda_m \neq \lambda_n$ for $m \neq n$ and l_n is the multiplicity of λ_n.

Note that l_n in (5.23) is not only the order of the zero of $D(\lambda)$ at λ_n, but also the multiplicity of λ_n. The entire function $D(\lambda)$ in the present case is of genus zero.

Problems

2.1 Establish the convergence results in (5.14) and (5.15).

2.2 Show that the three functions in (5.20) are all positive definite.

2.3 Establish the relation in (5.22). Show also that

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} = \int_0^1 \int_0^1 K^2(s,t)dsdt\,.$$

5.3. The c.f. of the quadratic Brownian functional

We now proceed to obtain the c.f. of

$$S = \int_0^1 \int_0^1 K(s,t)dw(s)dw(t)\,, \tag{5.24}$$

where $K(s,t)$ is symmetric, continuous and nearly definite. For this purpose we can first show (Problem 3.1) that S is the same in the m.s. sense as

$$S = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left\{ \int_0^1 f_n(t) dw(t) \right\}^2 ,$$

where $\{\lambda_n\}$ is a sequence of eigenvalues of K and $\{f_n(t)\}$ is an orthonormal sequence of eigenfunctions.

Then, noting that

$$\left\{ \int_0^1 f_n(t) dw(t) \right\} \sim \mathrm{NID}(0,1) ,$$

and using the product expansion for $D(\lambda)$ in (5.23), we obtain the following theorem, which was first established by Anderson and Darling (1952).

Theorem 5.4. *Consider the statistic S in (5.24), where $K(s,t)$ is continuous, symmetric and nearly definite on $[0,1] \times [0,1]$. Then we have*

$$\begin{aligned} (5.25) \qquad E(e^{i\theta S}) &= E\left[\exp\left\{ i\theta \int_0^1 \int_0^1 K(s,t) dw(s) dw(t) \right\} \right] \\ &= E\left[\exp\left\{ i\theta \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left(\int_0^1 f_n(t) dw(t) \right)^2 \right\} \right] \\ &= \prod_{n=1}^{\infty} \left(1 - \frac{2i\theta}{\lambda_n} \right)^{-\frac{1}{2}} \\ &= (D(2i\theta))^{-\frac{1}{2}} , \end{aligned}$$

where λ_n is the eigenvalue of K repeated as many times as its multiplicity and $D(\lambda)$ is the FD of K.

As was mentioned before it is usually difficult to obtain eigenvalues explicitly. Thus the second last expression in (5.25) is not very useful. Note that this corresponds to the eigenvalue approach discussed in Chapter 1. Our concern here is the last expression in (5.25). The problem is how to obtain $D(\lambda)$, which is now discussed.

The function $D(\lambda)$ may be defined in several ways, among which are

$$(5.26) \qquad D(\lambda) = \lim_{T \to \infty} \left| I_T - \frac{\lambda}{T} K_T \right|$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n}{n!} \int_0^1 \cdots \int_0^1 K \begin{pmatrix} t_1 & \cdot & \cdot & \cdot & t_n \\ t_1 & \cdot & \cdot & \cdot & t_n \end{pmatrix} dt_1 \cdots dt_n$$

$$= \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right)^{l_n} .$$

It is certainly true that there are cases where $D(\lambda)$ can be computed easily following one of the above formulas. One such example is the function $K_3(s,t)$ defined in (5.20). We obtain (Problem 3.2) the FD $D(\lambda)$ of K_3 as

$$D(\lambda) = 1 - \lambda \int_0^1 g^2(t) dt \, , \tag{5.27}$$

which is also obtained by noting that

$$E \left[\exp \left\{ i\theta \left(\int_0^1 g(t) dw(t) \right)^2 \right\} \right] = \left(1 - 2i\theta \int_0^1 g^2(t) dt \right)^{-\frac{1}{2}} .$$

In general, however, the computation of $D(\lambda)$ via the above formulas is difficult.

An alternative method for obtaining the FD is demonstrated in Nabeya and Tanaka (1988, 1990a), which we now explain. We first present a set of sufficient conditions for a function of λ to be the FD.

Theorem 5.5. *Let $K(s,t)$ be continuous, symmetric and nearly definite on $[0,1] \times [0,1]$ and $\{\lambda_n\}$ a sequence of eigenvalues of K. Suppose that $\tilde{D}(\lambda)$ is an entire function of λ with $\tilde{D}(0) = 1$. Then $\tilde{D}(\lambda)$ becomes the FD of K if*

- i) *every zero of $\tilde{D}(\lambda)$ is an eigenvalue of K, and in turn every eigenvalue of K is a zero of $\tilde{D}(\lambda)$;*
- ii) *$\tilde{D}(\lambda)$ can be expanded as*

$$\tilde{D}(\lambda) = \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right)^{l_n} , \tag{5.28}$$

where l_n is equal to the multiplicity of λ_n.

A word may be in order. If $\tilde{D}(\lambda)$ is an entire function with $\tilde{D}(0) = 1$, so is $\tilde{D}^2(\lambda)$, for example. The zero of $\tilde{D}^2(\lambda)$ at λ_n, however, is of order $2l_n$, while the multiplicity of λ_n is l_n. Thus $\tilde{D}^2(\lambda)$ is not the FD of K.

To obtain a candidate $\tilde{D}(\lambda)$ for the FD of K we work with a differential equation with some boundary conditions equivalent to the integral equation (5.10). As an illustration let us take up $K(s,t) = 1 - \max(s,t)$, which is positive definite, and consider

$$\begin{aligned} (5.29) \qquad f(t) &= \lambda \int_0^1 [1 - \max(s,t)]\, f(s)ds \\ &= \lambda \left[-t \int_0^t f(s)ds - \int_t^1 sf(s)ds + \int_0^1 f(s)ds \right]. \end{aligned}$$

By differentiation we have

$$f'(t) = -\lambda \int_0^t f(s)ds, \qquad f''(t) = -\lambda f(t)\,.$$

Then it can be shown (Problem 3.3) that the integral equation (5.29) is equivalent to the following differential equation with two boundary conditions :

$$(5.30) \qquad f''(t) + \lambda f(t) = 0\,, \qquad f(1) = f'(0) = 0\,.$$

Here the choice of boundary conditions is somewhat arbitrary as far as they are linearly independent, but the simpler the better, as is recognized shortly. The general solution to (5.30) is given by

$$(5.31) \qquad f(t) = c_1 \cos\sqrt{\lambda}\, t + c_2 \sin\sqrt{\lambda}\, t\,, \qquad f(1) = f'(0) = 0\,,$$

where c_1 and c_2 are arbitrary constants. From the boundary conditions $f(1) = f'(0) = 0$ we have the following homogeneous equation on $c = (c_1, c_2)'$:

$$(5.32) \qquad \begin{pmatrix} \cos\sqrt{\lambda} & \sin\sqrt{\lambda} \\ 0 & \sqrt{\lambda} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow M(\lambda)c = 0\,.$$

The eigenfunction $f(t)$ in (5.31) must be nonvanishing, which occurs only when $c \neq 0$. Thus (5.32) implies that $|M(\lambda)| = \sqrt{\lambda}\cos\sqrt{\lambda} = 0$. Then $\lambda(\neq 0)$ is an eigenvalue if and only if $\cos\sqrt{\lambda} = 0$. We therefore obtain $\tilde{D}(\lambda) = \cos\sqrt{\lambda}$ with $\tilde{D}(0) = 1$ as a candidate for the FD of $K(s,t) = 1 - \max(s,t)$. The condition i) in Theorem 5.5 has now been established.

We proceed to establish ii) in the same theorem. From (5.32) we have $c_2 = 0$ so that $f(t) = c_1 \cos\sqrt{\lambda} t$ with $c_1 \neq 0$. Thus the multiplicity of every eigenvalue is unity.

Since $\tilde{D}(\lambda) = \cos\sqrt{\lambda}$ can be expanded as in (5.28) with $\lambda_n = \left(\left(n - \frac{1}{2}\right)\pi\right)^2$ and $l_n = 1$, every zero of $\tilde{D}(\lambda)$ is of order unity, which is equal to the multiplicity of each eigenvalue. Therefore $\tilde{D}(\lambda)$ is really the FD of K.

Theorem 5.4 now yields

$$E\left[\exp\left\{i\theta\int_0^1\int_0^1 (1-\max(s,t))dw(s)dw(t)\right\}\right] = \left(\cos\sqrt{2i\theta}\right)^{-\frac{1}{2}}\,, \tag{5.33}$$

which was formally presented in Section 1 of Chapter 1.

Similarly, if $K(s,t) = \min(s,t) - st$, we can show (Problem 3.4) that the associated FD is given by $\sin\sqrt{\lambda}/\sqrt{\lambda}$ so that $\lambda_n = n^2\pi^2$ and

$$E\left[\exp\left\{i\theta\int_0^1\int_0^1 (\min(s,t)-st)dw(s)dw(t)\right\}\right] = \left(\frac{\sin\sqrt{2i\theta}}{\sqrt{2i\theta}}\right)^{-\frac{1}{2}}\,, \tag{5.34}$$

which was formally presented in Section 2 of Chapter 1.

In the above examples it was quite easy to obtain the FD's following Theorem 5.5. The eigenvalues can also be given explicitly. In general, however, we need some effort in verifying the condition ii) in the theorem together with the determination of the multiplicity l_n. As for the multiplicity we have the following theorem, which describes nothing but the dimension of a null space in the theory of matrices.

Theorem 5.6. *Suppose that the integral equation (5.10) is equivalent to a differential equation with some boundary conditions. Suppose further that the latter is equivalent to*

$$f(t) = c_1\phi_1(t) + \cdots + c_r\phi_r(t)\,,$$

$$M(\lambda)c = 0\,,$$

where ϕ_j's are linearly independent, continuous functions, while $M(\lambda)$ is the $r \times r$ coefficient matrix of the system of linear homogeneous equations in $c = (c_1, \cdots, c_r)'$. Then the multiplicity l_n of the eigenvalue λ_n is given by

$$l_n = r - \operatorname{rank}(M(\lambda_n))\,.$$

As an application of Theorem 5.6, consider the statistic S in (5.24) with

$$K(s,t) = \frac{1}{4}\left[1 - 2\,|s-t|\,\right], \tag{5.35}$$

which is shown to be positive definite (Problem 3.5). This kernel appears in connection with Lévy's stochastic area, as was shown in (5.5). We first have (Problem 3.6) that the integral equation (5.10) is equivalent to

$$f''(t) + \lambda f(t) = 0\,, \quad f(0) + f(1) = 0\,, \quad f'(0) + f'(1) = 0\,, \tag{5.36}$$

so that we are led to the homogeneous equation $M(\lambda)c = 0$, where

$$\begin{aligned} |M(\lambda)| &= \begin{vmatrix} 1 + \cos\sqrt{\lambda} & \sin\sqrt{\lambda} \\ -\sqrt{\lambda}\sin\sqrt{\lambda} & \sqrt{\lambda}(1+\cos\sqrt{\lambda}) \end{vmatrix} \\ &= 2\sqrt{\lambda}(1+\cos\sqrt{\lambda}) = 4\sqrt{\lambda}\left(\cos\frac{\sqrt{\lambda}}{2}\right)^2 . \end{aligned}$$

Then $\lambda(\neq 0)$ is an eigenvalue if and only if $|M(\lambda)| = 0$. Thus we obtain $\tilde{D}(\lambda) = \left(\cos\frac{\sqrt{\lambda}}{2}\right)^2$ as a candidate for the FD of K. Since $\mathrm{rank}(M(\lambda_n)) = 0$ for every eigenvalue $\lambda_n = ((2n-1)\pi)^2$, $(n = 1, 2, \cdots)$, the multiplicity of each eigenvalue is two. In fact, for $K(s,t)$ in (5.35), we have

$$\int_0^1 K(t,t)dt = \frac{1}{4}\,,$$

while

$$\sum_{n=1}^{\infty} \frac{1}{((2n-1)\pi)^2} = \frac{1}{8}\,,$$

and thus the equality in (5.22) holds with the multiplicity two. Since it is known that

$$\left(\cos\frac{\sqrt{\lambda}}{2}\right)^2 = \prod_{n=1}^{\infty}\left(1 - \frac{\lambda}{((2n-1)\pi)^2}\right)^2 ,$$

$\tilde{D}(\lambda)$ is ensured to be the FD of K and

$$\begin{aligned} E(e^{i\theta S}) &= E\left[\exp\left\{i\theta\int_0^1\int_0^1 \frac{1}{4}\left[1 - 2\left|s-t\right|\right]dw(s)dw(t)\right\}\right] \\ &= (\tilde{D}(2i\theta))^{-\frac{1}{2}} = \left(\cos\frac{\sqrt{2i\theta}}{2}\right)^{-1} . \end{aligned} \tag{5.37}$$

The c.f. of Lévy's stochastic area defined in (5.4) as

$$S_2 = \frac{1}{2}\int_0^1 [w_1(t)dw_2(t) - w_2(t)dw_1(t)]$$

can be easily derived from (5.37). Since

$$S_2 \,|\, \{w_1(t)\} \sim \mathrm{N}\left(0, \int_0^1 \int_0^1 \frac{1}{4}\left[1 - 2\,|s-t|\,\right] dw_1(s) dw_1(t)\right) ,$$

(5.37) yields

$$\begin{aligned} (5.38) \qquad E(e^{i\theta S_2}) &= E\left[\exp\left\{-\frac{\theta^2}{2}\int_0^1\int_0^1 \frac{1}{4}\left[1-2\,|s-t|\,\right] dw(s)dw(t)\right\}\right] \\ &= (\tilde{D}(-\theta^2))^{-\frac{1}{2}} = \left(\cosh\frac{\theta}{2}\right)^{-1} , \end{aligned}$$

which was earlier obtained in Section 4 of Chapter 1 by the eigenvalue approach and in Section 3 of Chapter 4 by the stochastic process approach.

Another case where the multiplicity of each eigenvalue is two can be found in Watson (1961), who suggested a goodness-of-fit test statistic on a circle with

$$(5.39) \qquad K(s,t) = \min(s,t) - \frac{1}{2}(s+t) + \frac{1}{2}(s-t)^2 + \frac{1}{12} .$$

In fact we can show (Problem 3.7) that the FD $D(\lambda)$ of K in (5.39) is given by

$$(5.40) \qquad D(\lambda) = \left(\sin\frac{\sqrt{\lambda}}{2} \Big/ \frac{\sqrt{\lambda}}{2}\right)^2 .$$

Thus it holds that

$$\begin{aligned} &E\left[\exp\left\{i\theta\int_0^1\int_0^1 \left(\min(s,t) - \frac{1}{2}(s+t) + \frac{1}{2}(s-t)^2 + \frac{1}{12}\right) dw(s)dw(t)\right\}\right] \\ &= \left(\sin\sqrt{\frac{i\theta}{2}} \Big/ \sqrt{\frac{i\theta}{2}}\right)^{-1} . \end{aligned}$$

Cases of multiplicities greater than unity may be rare. The most important is the case where the multiplicity is equal to unity for each eigenvalue. Then every zero of the candidate function $\tilde{D}(\lambda)$ must be of order unity. The infinite product expansion under such a circumstance is given by the following theorem (Whittaker and Watson (1958, p.137)).

Theorem 5.7. *Let $h(z)$ be an entire function with $h(0) = 1$ and have simple zeros at the points $a_1, a_2, \cdots$, where $\lim_{n\to\infty} |a_n| = \infty$. Suppose that there is a sequence*

$\{C_m\}$ of simple closed curves such that $h'(z)/h(z)$ is bounded on C_m as $m \to \infty$. Then $h(z)$ can be expanded as

$$h(z) = \exp\{h'(0)z\} \prod_{n=1}^{\infty} \left\{ \left(1 - \frac{z}{a_n}\right) \exp\left(\frac{z}{a_n}\right) \right\} .$$

As an application of this theorem let us consider a nearly definite kernel

$$K(s,t) = 1 - \max(s,t) + b\,, \tag{5.41}$$

where b is any nonzero constant. We obtain

$$f''(t) + \lambda f(t) = 0\,, \quad f(1) = \lambda b \int_0^1 f(s)ds\,, \quad f'(0) = 0\,, \tag{5.42}$$

the set of which is equivalent to the original integral equation (5.10) (Problem 3.8). Solving for $f(t)$ we have, from the two boundary conditions, the homogeneous equation $M(\lambda)c = 0$, where

$$M(\lambda) = \begin{pmatrix} \cos\sqrt{\lambda} - b\sqrt{\lambda}\sin\sqrt{\lambda} & \sin\sqrt{\lambda} + b\sqrt{\lambda}(\cos\sqrt{\lambda} - 1) \\ 0 & \sqrt{\lambda} \end{pmatrix} .$$

Thus we obtain, as a candidate for the FD,

$$\tilde{D}(\lambda) = \cos\sqrt{\lambda} - b\sqrt{\lambda}\sin\sqrt{\lambda}\,, \qquad \tilde{D}(0) = 1\,. \tag{5.43}$$

Evidently $\operatorname{rank}(M(\lambda_n)) = 1$ for each eigenvalue λ_n so that, by Theorem 5.6, the multiplicity of λ_n is unity for all n.

Let us put $z = \sqrt{\lambda}$ and consider

$$h(z) = \tilde{D}(\lambda) = \cos z - bz\sin z\,, \tag{5.44}$$

which is an even entire function with $h(0) = 1$ whose zeros are all simple (Problem 3.9). Then we can define the zeros of $h(z)$ by $\pm a_1,\ \pm a_2, \cdots$, where $\lim_{n\to\infty}|a_n| = \infty$. Let C_m be the square in the complex plane with vertices $\left(2m + \frac{1}{2}\right)\pi(\pm 1 \pm i)$, $m = 1, 2, \cdots$. Then it is seen that

$$\frac{h'(z)}{h(z)} = \frac{-bz\cos z - (b+1)\sin z}{\cos z - bz\sin z}$$

is bounded on each side of C_m as $m \to \infty$. Note that $h'(z)/h(z)$ is not bounded on squares with vertices $m\pi(\pm 1 \pm i)$ since it takes the value $-bz$ at $z = m\pi$.

Using Theorem 5.7, we can expand the even function $h(z)$ with $h'(0) = 0$ as

$$\begin{aligned} h(z) &= \prod_{n=1}^{\infty} \left\{ \left(1 - \frac{z}{a_n}\right) \exp\left(\frac{z}{a_n}\right) \left(1 + \frac{z}{a_n}\right) \exp\left(-\frac{z}{a_n}\right) \right\} \\ &= \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{a_n^2}\right) = \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{a_n^2}\right) \\ &= \tilde{D}(\lambda)\,, \end{aligned}$$

from which we conclude that $\tilde{D}(\lambda)$ in (5.43) is the FD of K in (5.41). Note that $\{a_n^2\}$ is a sequence of eigenvalues of the symmetric kernel $K(s,t)$ so that each a_n^2 is real. It can be checked (Problem 3.10) that, when the value of b in (5.41) is nonnegative, every a_n^2 is positive, while all a_n^2's except one are positive when b is negative.

Some other examples of how to obtain the FD's of given kernels will be given in the next section.

Problems

3.1 Show that the following equality holds in the m.s. sense:

$$\int_0^1 \int_0^1 K(s,t) dw(s) dw(t) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left\{ \int_0^1 f_n(t) dw(t) \right\}^2 ,$$

where $K(s,t)$ is symmetric, continuous and nearly definite, while $\{\lambda_n\}$ is a sequence of eigenvalues of K and $\{f_n(t)\}$ is an orthonormal sequence of eigenfunctions.

3.2 Derive the FD of K_3 defined in (5.20) following one of the definitions of $D(\lambda)$ in (5.26).

3.3 Show that the integral equation (5.10) is equivalent to (5.30) when $K(s,t) = 1 - \max(s,t)$.

3.4 Derive the c.f. of S in (5.1) when $K(s,t) = \min(s,t) - st$, following the Fredholm approach.

3.5 Show that the kernel defined in (5.35) is positive definite.

3.6 Show that the integral equation (5.10) is equivalent to (5.36) when $K(s,t) = \frac{1}{4}[1 - 2|s-t|]$.

3.7 Prove that the FD of $K(s,t)$ in (5.39) is given by (5.40).

3.8 Show that the integral equation (5.10) is equivalent to (5.42) when $K(s,t) = 1 - \max(s,t) + b$.

3.9 Show that the zeros of $h(z) = \cos z - bz \sin z$ are all simple.

3.10 Show that all the eigenvalues of $K(s,t)$ in (5.41) are positive when b is nonnegative, while only one eigenvalue is negative when b is negative.

5.4. Various Fredholm determinants

We continue to derive the c.f.'s of various quadratic functionals of the Brownian motion. The examples considered here are thought to be those which cannot be dealt with easily by the stochastic process or eigenvalue approach.

Let us first consider the statistic S_1 defined in (5.3). In particular we consider

$$U_1 = \int_0^1 (t^m w(t))^2 dt = \int_0^1 \int_0^1 \frac{1}{2m+1} \left[1 - (\max(s,t))^{2m+1}\right] dw(s)dw(t)\,, \tag{5.45}$$

where $m > -\frac{1}{2}$. Let us show that

$$\mathcal{L}(U_1) = \mathcal{L}\left(\int_0^1 \int_0^1 s^m t^m \min(s,t) dw(s)dw(t)\right) . \tag{5.46}$$

For this purpose we first note that

$$V\left(\int_0^1 t^m w(t) g(t) dt\right) = \int_0^1 \int_0^1 s^m t^m \min(s,t) g(s) g(t) ds dt \geq 0$$

for any continuous function $g(t)$, which implies that $s^m t^m \min(s,t)$ is positive definite. Thus we can define

$$Z(t) = \sum_{n=1}^{\infty} \frac{f_n(t)}{\sqrt{\lambda_n}} Z_n\,,$$

where $\{Z_n\} \sim \mathrm{NID}(0,1)$ and each $\lambda_n\,(>0)$ is the eigenvalue of $s^m t^m \min(s,t)$, while $f_n(t)$ is an orthonormal eigenfunction corresponding to λ_n. Since it can be easily checked that any finite dimensional distribution of $Z(t)$ is the same as $t^m w(t)$ and the finite-dimensional sets form a determining class (see Section 3.2 of Chapter 3), it follows from Mercer's theorem that

$$\begin{aligned}\mathcal{L}(U_1) &= \mathcal{L}\left(\int_0^1 Z^2(t)dt\right)\\ &= \mathcal{L}\left(\sum_{n=1}^{\infty}\frac{1}{\lambda_n}Z_n^2\right) = \mathcal{L}\left(\int_0^1\int_0^1 s^m t^m \min(s,t)dw(s)dw(t)\right)\,.\end{aligned}$$

The above distributional equivalence for $m=0$ was earlier presented in (1.11). Note that the kernel appearing in (5.45) is positive definite (Problem 4.1). The statistic U_1 was dealt with by MacNeill (1974) and Nabeya and Tanaka (1988) in connection with testing for parameter constancy. MacNeill (1974) assumed $m>-1$ rather than $m>-\frac{1}{2}$, although the kernel is not continuous when $-1<m\le-\frac{1}{2}$. Here we do not go into such a complexity, but continue to assume that the kernel is continuous so that $m>-\frac{1}{2}$.

We can show (Problem 4.2) that the integral equation (5.10) with $K(s,t) = K_1(s,t) = (1-(\max(s,t))^{2m+1})/(2m+1)$ is equivalent to

$$\text{(5.47)}\qquad f''(t)-\frac{2m}{t}f'(t)+\lambda t^{2m}f(t)=0\,,\quad \lim_{t\to 0}\frac{f'(t)}{t^{2m}}=0\,,\quad f(1)=0\,.$$

The differential equation in (5.47) is a special case of Bessel's equation :

$$\text{(5.48)}\qquad y''(x)-\frac{2\alpha-1}{x}y'(x)+\left(\beta^2\gamma^2x^{2\gamma-2}+\frac{\alpha^2-\nu^2\gamma^2}{x^2}\right)y(x)=0\,.$$

It is known (Abramowitz and Stegun (1972)) that (5.48) has the general solution:

$$\text{(5.49)}\qquad y(x)=\begin{cases} x^{\alpha}\left(AJ_\nu(\beta x^\gamma)+BY_\nu(\beta x^\gamma)\right)\,, & (\nu:\ \text{integer})\,,\\ x^{\alpha}\left(AJ_\nu(\beta x^\gamma)+BJ_{-\nu}(\beta x^\gamma)\right)\,, & (\nu:\ \text{noninteger})\,,\end{cases}$$

where $J_\nu(z)$ is the Bessel function of the first kind defined by

$$\text{(5.50)}\qquad J_\nu(z)=\left(\frac{z}{2}\right)^{\nu}\sum_{k=0}^{\infty}\frac{\left(-\frac{z^2}{4}\right)^k}{k!\;\Gamma(\nu+k+1)}\,,$$

while $Y_\nu(z)$ is the Bessel function of the second kind (also called Weber's function) defined by

$$Y_\nu(z) = \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}\,. \tag{5.51}$$

On the basis of (5.48) and (5.49) we obtain

$$\begin{aligned} f(t) &= t^{\frac{2m+1}{2}} \left\{ c_1 J_\nu\left(\frac{\sqrt{\lambda}}{m+1}\, t^{m+1}\right) + c_2 J_{-\nu}\left(\frac{\sqrt{\lambda}}{m+1}\, t^{m+1}\right) \right\} \\ &= c_1 \left(\frac{\sqrt{\lambda}}{2(m+1)}\right)^{\nu} t^{2m+1} \sum_{k=0}^{\infty} \frac{\left(-\frac{\lambda t^{2(m+1)}}{4(m+1)^2}\right)^k}{k!\,\Gamma(\nu+k+1)} \\ &\qquad + c_2 \left(\frac{\sqrt{\lambda}}{2(m+1)}\right)^{-\nu} \sum_{k=0}^{\infty} \frac{\left(-\frac{\lambda t^{2(m+1)}}{4(m+1)^2}\right)^k}{k!\,\Gamma(-\nu+k+1)}\,, \end{aligned}$$

where $\nu = (2m+1)/(2(m+1))$. Note that ν cannot be an integer when $m > -\dfrac{1}{2}$. From the two boundary conditions in (5.47) it follows (Problem 4.3) that $M(\lambda)c = 0$, where $c = (c_1\,, c_2)'$ and

$$M(\lambda) = \begin{pmatrix} \left(\dfrac{\sqrt{\lambda}}{2(m+1)}\right)^{\nu} \dfrac{2m+1}{\Gamma(\nu+1)} & 0 \\ J_\nu\!\left(\dfrac{\sqrt{\lambda}}{m+1}\right) & J_{-\nu}\!\left(\dfrac{\sqrt{\lambda}}{m+1}\right) \end{pmatrix}. \tag{5.52}$$

Then we obtain, from (5.50),

$$\tilde{D}_1(\lambda) = \Gamma(-\nu+1)\, J_{-\nu}\left(\frac{\sqrt{\lambda}}{m+1}\right) \Big/ \left(\frac{\sqrt{\lambda}}{2(m+1)}\right)^{-\nu} \tag{5.53}$$

as a candidate for the FD of K_1 with $\tilde{D}_1(0) = 1$. It is clear that $\mathrm{rank}(M(\lambda_n)) = 1$ for each eigenvalue λ_n so that the multiplicity of every eigenvalue is unity. Since it is known (Watson (1958, p.498)) that

$$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^{\nu}}{\Gamma(\nu+1)} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{a_n^2}\right)\,,$$

where $a_1 < a_2 < \cdots$ are the positive zeros of $J_\nu(z)$, $\tilde{D}_1(\lambda)$ can be expanded as

$$\tilde{D}_1(\lambda) = \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{(m+1)^2 a_n^2}\right)\,,$$

which implies that all the zeros of $\tilde{D}_1(\lambda)$ are positive and simple. Thus we have verified that $\tilde{D}_1(\lambda)$ is the FD of K_1 and the c.f. of U_1 in (5.44) is given by $(\tilde{D}_1(2i\theta))^{-\frac{1}{2}}$.

It may be noted that, when $m = 0$ so that $\nu = \dfrac{1}{2}$, $\tilde{D}_1(\lambda)$ reduces to $\cos\sqrt{\lambda}$ since

$$J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\cos z\,.$$

Closely related to the statistic U_1 is

$$U_2 = \int_0^1 t^m w(t)dw(t)\,, \qquad (m \geq 0)\,, \tag{5.54}$$

which is a special case of S_3 defined in (5.6) with $g(t) = t^m$. It follows from (5.7) that

$$U_2 = \int_0^1\int_0^1 \frac{1}{2}(\max(s,t))^m dw(s)dw(t) - \frac{1}{2(m+1)}\,. \tag{5.55}$$

Note here that the kernel $K_2(s,t) = (\max(s,t))^m/2$ is nearly definite since it is the sum of a negative definite kernel $-(1-(\max(s,t))^m)/2$ and a degenerate kernel $\dfrac{1}{2}$.

It can be checked that

$$f''(t) - \frac{m-1}{t}f'(t) - \frac{\lambda m}{2}t^{m-1}f(t) = 0\,, \quad \lim_{t\to 0}\frac{f'(t)}{t^{m-1}} = 0\,, \quad f'(1) = mf(1) \tag{5.56}$$

are equivalent to the integral equation (5.10) with $K(s,t) = K_2(s,t)$. Then we obtain (Problem 4.4), as the FD of K_2,

$$\begin{aligned} D_2(\lambda) &= \Gamma\left(\frac{1}{m+1}\right)\left(-\frac{m+1}{m}\right)J_{\nu-1}\left(\frac{\sqrt{-2\lambda m}}{m+1}\right)\Bigg/\left(\frac{\sqrt{-2\lambda m}}{2(m+1)}\right)^{\nu-1} \\ &= \Gamma(\nu)J_{\nu-1}\left(\frac{\sqrt{-2\lambda m}}{m+1}\right)\Bigg/\left(\frac{\sqrt{-2\lambda m}}{2(m+1)}\right)^{\nu-1}\,, \end{aligned} \tag{5.57}$$

where $\nu = -m/(m+1)$. Thus the c.f. of U_2 in (5.55) is given by $(D_2(2i\theta))^{-\frac{1}{2}}\exp\{-i\theta/(2(m+1))\}$. The first expression in (5.57) is more suitable for the computational purpose. It may be noted (Problem 4.5) that, when $m = 0$, $D_2(\lambda)$ reduces to $1 - \dfrac{\lambda}{2}$, as should be.

We next consider a quadratic functional of the Brownian bridge. Let us put

$$\begin{aligned} U_3 &= \int_0^1 \left[t^m\{w(t) - tw(1)\}\right]^2 dt \\ &= \int_0^1\int_0^1 \left[\frac{1}{2m+1}\left(1-(\max(s,t))^{2m+1}\right) - \frac{2-s^{2m+2}-t^{2m+2}}{2m+2}\right. \\ &\qquad \left. + \frac{1}{2m+3}\right]dw(s)dw(t)\,, \end{aligned} \tag{5.58}$$

where $m > -\dfrac{1}{2}$. Because of the same reasoning as in establishing (5.46) we also have

$$\mathcal{L}(U_3) = \mathcal{L}\left(\int_0^1 \int_0^1 s^m t^m [\min(s,t) - st]\, dw(s) dw(t)\right) . \tag{5.59}$$

The above distributional equivalence for $m = 0$ was earlier presented in (1.18). Proceeding in the same way as above we obtain (Problem 4.6), as the FD of $K_3(s,t) = s^m t^m (\min(s,t) - st)$,

$$D_3(\lambda) = \Gamma\left(\frac{2m+3}{2(m+1)}\right) J_{\frac{1}{2(m+1)}}\left(\frac{\sqrt{\lambda}}{m+1}\right) \Big/ \left(\frac{\sqrt{\lambda}}{2(m+1)}\right)^{\frac{1}{2(m+1)}} . \tag{5.60}$$

It can be checked that, when $m = 0$, $D_3(\lambda)$ reduces to $D_3(\lambda) = \sin\sqrt{\lambda}/\sqrt{\lambda}$ since

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\, \sin z\, .$$

It may also be noted that the statistic on the right side of (5.59) is a special case of

$$W^2 = \int_0^1 \int_0^1 \sqrt{\psi(s)}\sqrt{\psi(t)}\, [\min(s,t) - st]\, dw(s) dw(t)\, , \tag{5.61}$$

which was discussed by Anderson and Darling (1952) in connection with goodness of fit tests. The so-called Anderson-Darling statistic is the one with $\psi(t) = 1/(t(1-t))$ and the c.f. of W^2 in that case was obtained in Anderson and Darling (1952) by the eigenvalue approach, although the kernel is not continuous at $(s,t) = (0,0)$ or (1,1). The statistic U_3 in (5.58) or (5.59) is W^2 in (5.61) with $\psi(t) = t^{2m}$. If we take m to be negative, then this may be an alternative to the Anderson-Darling statistic.

We also point out (Problem 4.7) that

$$\mathcal{L}(U_3) = \mathcal{L}\left(\int_0^1 \int_0^1 \frac{1}{2m+1}\left[(\min(s,t))^{\frac{1}{2m+1}} - (st)^{\frac{1}{2m+1}}\right] dw(s) dw(t)\right) . \tag{5.62}$$

A slightly modified version of U_3 is

$$\begin{aligned} U_4 \;&=\; \int_0^1 \left[t^m \left\{w(t) - (2m+1)\int_0^1 s^{2m} w(s) ds\right\}\right]^2 dt \\ &=\; \int_0^1 \int_0^1 \frac{1}{2m+1}\left[(\min(s,t))^{2m+1} - (st)^{2m+1}\right] dw(s) dw(t)\, , \end{aligned} \tag{5.63}$$

where $m > -\dfrac{1}{2}$. It is now an easy matter to obtain (Problem 4.8) the FD of $K_4(s,t) = ((\min(s,t))^{2m+1} - (st)^{2m+1})/(2m+1)$ as

$$D_4(\lambda) = \Gamma\left(\frac{4m+3}{2(m+1)}\right) J_{\frac{2m+1}{2(m+1)}}\left(\frac{\sqrt{\lambda}}{m+1}\right) \Big/ \left(\frac{\sqrt{\lambda}}{2(m+1)}\right)^{\frac{2m+1}{2(m+1)}} . \tag{5.64}$$

The statistic U_4 in (5.63) was dealt with in Nabeya and Tanaka (1988).

Nabeya and Tanaka (1988) also suggested a method for computing c.f.'s of quadratic functionals of the Brownian motion whose kernel is given by

$$K(s,t) = \min(s,t) + \sum_{k=1}^{r} \xi_k(s)\psi_k(t) , \tag{5.65}$$

where

i) $\xi_k(s)$ and $\psi_k(t)$ $(k = 1, \cdots, r)$ are continuous and each set is linearly independent in the space $C[0,1]$;

ii) $\psi_k(t)$ $(k = 1, \cdots, r)$ are twice continuously differentiable and $\psi_k''(t)$ for $k = 1, \cdots, q$ are linearly independent, whereas $\psi_k''(t) = 0$ for $k = q+1, \cdots, r$ with $0 \le r - q \le 2$.

The condition ii) above allows for the space of linear combinations of $\psi_k(t)$ to contain a constant and/or a linear function.

We now consider

$$f(t) = \lambda \int_0^1 \left[\min(s,t) + \sum_{k=1}^{r} \xi_k(s)\psi_k(t)\right] f(s)ds , \tag{5.66}$$

from which we obtain the nonhomogeneous differential equation

$$f''(t) + \lambda f(t) = \lambda \sum_{k=1}^{q} a_k \psi_k''(t) , \tag{5.67}$$

with the boundary conditions

$$f(0) = \lambda \sum_{k=1}^{r} a_k \psi_k(0) , \tag{5.68}$$

$$f'(1) = \lambda \sum_{k=1}^{r} a_k \psi_k'(1) , \tag{5.69}$$

where

$$a_k = \int_0^1 \xi_k(s) f(s) ds\,, \qquad (k = 1, \cdots, r)\,. \tag{5.70}$$

The associated FD is obtained as follows, by modifying the technique of Kac, Kiefer and Wolfowitz (1955). For given $\lambda (\neq 0)$, the general solution to (5.67) is

$$f(t) = c_1 \cos \sqrt{\lambda}\, t + c_2 \sin \sqrt{\lambda}\, t + \sum_{k=1}^{q} a_k g_k(t)\,, \tag{5.71}$$

where $g_k(t)$ is a special solution of

$$g_k''(t) + \lambda g_k(t) = \lambda \psi_k''(t)\,. \tag{5.72}$$

Substituting $f(t)$ from (5.71), we regard (5.68), (5.69) and (5.70) as a system of $r+2$ linear homogeneous equations $M(\lambda)c = 0$ in $c = (a_1, \cdots, a_r,\, c_1,\, c_2)'$. Then it can be shown that $\lambda(\neq 0)$ is an eigenvalue of K if and only if the system has a nontrivial solution so that we are led to compute $|M(\lambda)|$ to obtain a candidate for the FD.

As an illustration let us consider

$$U_5 = \int_0^1 \int_0^1 \left[\min(s,t) + \frac{t}{16}(9s^5 - 25s) - \frac{9}{16} t^5 (s^5 - s) \right] dw(s) dw(t)\,, \tag{5.73}$$

which is discussed in Nabeya and Tanaka (1988), where the kernel, denoted as $K_5(s,t)$, is positive definite and may be rewritten as in (5.65) with $q = 1$ and $r = 2$ by putting

$$\begin{aligned} \xi_1(s) &= -s^5 + s\,, &\qquad \xi_2(s) &= 9s^5 - 25s\,, \\ \psi_1(t) &= \frac{9}{16} t^5\,, &\qquad \psi_2(t) &= \frac{t}{16}\,. \end{aligned}$$

We obtain

$$f''(t) + \lambda f(t) = \frac{45}{4} \lambda a_1 t^3 \tag{5.74}$$

with the boundary conditions

$$f(0) = 0\,, \qquad f'(1) = \frac{\lambda}{16}(45 a_1 + a_2)\,, \tag{5.75}$$

where

$$a_1 = \int_0^1 (-s^5 + s) f(s) ds\,, \qquad a_2 = \int_0^1 (9s^5 - 25s) f(s) ds\,. \tag{5.76}$$

The general solution to (5.74) is given by

$$f(t) = c_1 \cos\sqrt{\lambda}\,t + c_2 \sin\sqrt{\lambda}\,t + \frac{45a_1}{4}\left(t^3 - \frac{6}{\lambda}t\right) . \tag{5.77}$$

Substituting this into (5.75) and (5.76) yields $M(\lambda)c = 0$, where $c = (a_1,\, a_2,\, c_1,\, c_2)'$ and

$$M(\lambda) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ M_{21}(\lambda) & -\dfrac{\lambda}{16} & -\sqrt{\lambda}\sin\sqrt{\lambda} & \sqrt{\lambda}\cos\sqrt{\lambda} \\ -\dfrac{90}{7\lambda} & 0 & M_{33}(\lambda) & M_{34}(\lambda) \\ M_{41}(\lambda) & -1 & M_{43}(\lambda) & M_{44}(\lambda) \end{pmatrix} \tag{5.78}$$

with

$$\begin{aligned}
M_{21}(\lambda) &= \frac{135}{4} - \frac{45}{16}\lambda - \frac{135}{2\lambda} , \\
M_{33}(\lambda) &= \frac{1}{\lambda^3}\left[-4(\lambda^2 - 15\lambda + 30)\cos\sqrt{\lambda} + 20\sqrt{\lambda}(\lambda - 6)\sin\sqrt{\lambda} + \lambda^2 - 120\right] , \\
M_{34}(\lambda) &= \frac{4}{\lambda^3}\left[-5\sqrt{\lambda}(\lambda - 1)\cos\sqrt{\lambda} - (\lambda^2 - 15\lambda + 30)\sin\sqrt{\lambda}\right] , \\
M_{41}(\lambda) &= -45 + \frac{3330}{7\lambda} , \\
M_{43}(\lambda) &= \frac{1}{\lambda^3}\Big[20(\lambda^2 - 27\lambda + 54)\cos\sqrt{\lambda} \\
&\qquad -4\sqrt{\lambda}(4\lambda^2 + 45\lambda - 270)\sin\sqrt{\lambda} - 25\lambda^2 + 1080\Big] , \\
M_{44}(\lambda) &= \frac{4}{\lambda^3}\left[\sqrt{\lambda}(4\lambda^2 + 45\lambda - 270)\cos\sqrt{\lambda} + 5(\lambda^2 - 27\lambda + 54)\sin\sqrt{\lambda}\right] .
\end{aligned}$$

Making use of the computerized algebra REDUCE we obtain, as a candidate for the FD,

$$\begin{aligned}
\tilde{D}_5(\lambda) &= |M(\lambda)|/\sqrt{\lambda} \\
&= \frac{1350}{\lambda^5}\left[\left(-\frac{\lambda^3}{6} + 2\lambda^2 - 6\lambda\right)\cos\sqrt{\lambda}\right. \\
&\qquad\qquad \left. + \sqrt{\lambda}\left(-\frac{\lambda^3}{42} + \frac{7\lambda^2}{10} - 4\lambda + 6\right)\sin\sqrt{\lambda}\right] .
\end{aligned} \tag{5.79}$$

Here every zero of $h(z) = \lambda^5\tilde{D}_5(\lambda)/1350$ with $z = \sqrt{\lambda}$ is real because nonzero $z^2 = \lambda$ is an eigenvalue of the positive definite kernel K_5. Let $a_1, a_2, \cdots$ be the positive zeros of $h(z)$. Then the rank of the 4×4 coefficient matrix $M(a_j^2)$ is 3, which implies that the multiplicity of every eigenvalue is unity. It can also be checked that every nonzero solution of $h(z) = 0$ is simple by showing that there exists no nonzero solution common to $h(z) = 0$ and $h'(z) = 0$. Moreover, the function $h^*(z) = 1350h(z)/z^{10} = \tilde{D}_5(\lambda)$ is even and analytic with $h^*(0) = 1$ and with the zeros $\pm a_1, \ \pm a_2, \cdots$. Since $h^{*\prime}(z)/h^*(z)$ is bounded on the square C_m with vertices $\left(2m + \frac{1}{2}\right)\pi(\pm 1 \pm i)$, we can verify that $\tilde{D}_5(\lambda)$ in (5.79) is the FD of K_5.

In the derivation of $\tilde{D}_5(\lambda)$ we have used the four conditions given in (5.75) and (5.76), which enforces us to compute the determinant of the 4×4 matrix $M(\lambda)$ in (5.78). In the present case, however, it is more convenient to use the boundary condition $f(1) = 0$, in stead of $f'(1) = \lambda(45a_1 + a_2)/16$ in (5.75). This enables us to dispense with the introduction of a_2 since the general solution $f(t)$ in (5.77) does not contain a_2. Then we can deal with a 3×3 matrix rather than the 4×4 matrix $M(\lambda)$ and we can arrive at the same result (Problem 4.9).

The same technique can be applied to obtain the c.f. of

$$U_6 = \int_0^1 \int_0^1 \left[\min(s,t) - st - \frac{2}{\pi^2} \sin \pi s \, \sin \pi t\right] dw(s)dw(t)\,. \tag{5.80}$$

Denoting as $K_6(s,t)$ the kernel appearing in (5.80) we can show that the integral equation (5.10) with $K = K_6$ is equivalent to

$$f''(t) + \lambda f(t) = 2\lambda a_1 \sin \pi t \tag{5.81}$$

with the boundary conditions $f(0) = f(1) = 0$ and

$$a_1 = \int_0^1 \sin \pi s f(s) ds\,.$$

When $\lambda \neq \pi^2$, the general solution to (5.81) is given by

$$f(t) = c_1 \cos \sqrt{\lambda}\, t + c_2 \sin \sqrt{\lambda}\, t - \frac{2a_1\lambda}{\pi^2 - \lambda} \sin \pi t$$

and the above three conditions yield $M(\lambda)c = 0$, where $c = (a_1, \, c_1, \, c_2)'$ and

$$M(\lambda) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \cos\sqrt{\lambda} & \sin\sqrt{\lambda} \\ \dfrac{-\pi^2}{\pi^2 - \lambda} & \dfrac{\pi(\cos\sqrt{\lambda} + 1)}{\pi^2 - \lambda} & \dfrac{\pi \sin\sqrt{\lambda}}{\pi^2 - \lambda} \end{pmatrix},$$

$$(5.82) \qquad |M(\lambda)| = -\frac{\pi^2}{\pi^2 - \lambda} \sin\sqrt{\lambda}\,.$$

Suppose that $\lambda = \pi^2$. Then the general solution to (5.81) is

$$f(t) = c_1 \cos \pi t + c_2 \sin \pi t - a_1 \pi t \cos \pi t\,.$$

The above three conditions yield $Nc = 0$, where $c = (a_1,\, c_1,\, c_2)'$ and

$$N = \begin{pmatrix} 0 & 1 & 0 \\ \pi & -1 & 0 \\ -\dfrac{3}{4} & 0 & \dfrac{1}{2} \end{pmatrix}, \qquad |N| = -\frac{\pi}{2} \neq 0\,.$$

Thus $\lambda = \pi^2$ is not an eigenvalue. Then we obtain, from (5.82),

$$(5.83) \qquad \begin{aligned} D_6(\lambda) &= \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} \Bigg/ \left(1 - \frac{\lambda}{\pi^2}\right) \\ &= \prod_{n=2}^{\infty} \left(1 - \frac{\lambda}{n^2\pi^2}\right) \end{aligned}$$

as the FD of $K_6(s,t) = \min(s,t) - st - \dfrac{2}{\pi^2} \sin \pi s\, \sin \pi t$; hence the c.f. of U_6 in (5.80) is given by $(D_6(2i\theta))^{-\frac{1}{2}}$.

The above result may be derived as follows. By Mercer's theorem (Theorem 5.2) it holds that

$$\min(s,t) - st = \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2} \sin n\pi s\, \sin n\pi t\,,$$

where $\{\sqrt{2} \sin n\pi t\}$ is an orthonormal sequence of eigenfunctions corresponding to $\lambda_n = n^2\pi^2$. Therefore

$$K_6(s,t) = \sum_{n=2}^{\infty} \frac{2}{n^2\pi^2} \sin n\pi s\, \sin n\pi t\,,$$

which yields $D_6(\lambda)$ in (5.83). The same reasoning can be applied to obtain the F.D. $D(\lambda)$ of

$$K(s,t) = \min(s,t) - st - \sum_{n=1}^{k} \frac{2}{n^2\pi^2} \sin n\pi s\, \sin n\pi t\,.$$

We have

$$
\begin{aligned}
D(\lambda) &= \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} \Bigg/ \prod_{n=1}^{k} \left(1 - \frac{\lambda}{n^2\pi^2}\right) \\
&= \prod_{n=k+1}^{\infty} \left(1 - \frac{\lambda}{n^2\pi^2}\right) .
\end{aligned}
$$

Let us consider a slightly different statistic :

$$
U_7 = \int_0^1 \int_0^1 \left[\min(s,t) - st - \frac{2}{\pi^2}\sin^2\pi s\,\sin^2\pi t\right] dw(s)dw(t) . \tag{5.84}
$$

Mercer's theorem is not helpful here, but, following the method demonstrated before, we can show (Nabeya (1992) and Problem 4.10) that $E(e^{i\theta U_7}) = (D_7(2i\theta))^{-\frac{1}{2}}$, where

$$
D_7(\lambda) = \left[\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} + \frac{1-\cos\sqrt{\lambda}}{\pi^2} \Bigg/ \left(1 - \frac{\lambda}{4\pi^2}\right)\right] \Bigg/ \left(1 - \frac{\lambda}{4\pi^2}\right) . \tag{5.85}
$$

Nabeya (1992) points out that the corresponding result earlier obtained by Darling (1955) is incorrect. It can be checked that $\lambda = 4\pi^2$ is an eigenvalue although the factor $1 - (\lambda/(4\pi^2))$ appears in the denominator of $D_7(\lambda)$.

Similarly, if we consider an extended version:

$$
\begin{aligned}
U_8 &= \int_0^1 \int_0^1 \Big[\min(s,t) - st - \frac{2}{\pi^2}\sin^2\pi s\,\sin^2\pi t \\
&\qquad\qquad - \frac{1}{2\pi^2}\sin 2\pi s\,\sin 2\pi t\Big] dw(s)dw(t) ,
\end{aligned} \tag{5.86}
$$

we can show (Nabeya (1992) and Problem 4.11) that $E(e^{i\theta U_8}) = (D_8(2i\theta))^{-\frac{1}{2}}$, where

$$
\begin{aligned}
D_8(\lambda) &= D_7(\lambda) \Bigg/ \left(1 - \frac{\lambda}{4\pi^2}\right) \\
&= \left[\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} + \frac{1-\cos\sqrt{\lambda}}{\pi^2} \Bigg/ \left(1 - \frac{\lambda}{4\pi^2}\right)\right] \Bigg/ \left(1 - \frac{\lambda}{4\pi^2}\right)^2 .
\end{aligned} \tag{5.87}
$$

It can be checked that $\lambda = 4\pi^2$ is not an eigenvalue.

As a final example we deal with the integrated Brownian motion. Let us consider the statistic defined in (5.8). When $g = 1$, the statistic becomes

$$
\begin{aligned}
U_9 &= \int_0^1 F_1^2(t)dt \\
&= \int_0^1 \int_0^1 \left[\int_{\max(s,t)}^1 (u-s)(u-t)du\right] dw(s)dw(t) ,
\end{aligned} \tag{5.88}
$$

where $\{F_1(t)\}$ is the one-fold integrated Brownian motion defined by

$$F_1(t) = \int_0^t w(s)ds\,.$$

The kernel appearing in (5.88), which we denote as $K_9(s,t)$, has the expression

$$K_9(s,t) = \frac{1}{6}(1-t)^2(t+2-3s)$$

for $s \le t$.

The integral equation (5.10) with $K = K_9$ is equivalent to

$$f^{(4)}(t) - \lambda f(t) = 0\,, \quad f(1) = f'(1) = f''(0) = f'''(0) = 0\,.$$

The differential equation has the general solution

$$f(t) = c_1 e^{At} + c_2 e^{-At} + c_3 e^{iAt} + c_4 e^{-iAt}\,, \quad A = \lambda^{\frac{1}{4}}\,,$$

for arbitrary constants c_1 through c_4. The four boundary conditions above yield $M(\lambda)c = 0$, where $c = (c_1,\, c_2,\, c_3,\, c_4)'$ and

$$M(\lambda) = \begin{pmatrix} e^{A} & e^{-A} & e^{iA} & e^{-iA} \\ e^{A} & -e^{-A} & ie^{iA} & -ie^{-iA} \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -i & i \end{pmatrix},$$

$$|M(\lambda)| = 8i(1+\cos A \cosh A)\,.$$

Then we obtain, as the FD of K_9,

$$D_9(\lambda) = \frac{1}{2}\left(1 + \cos\lambda^{\frac{1}{4}} \cosh\lambda^{\frac{1}{4}}\right)\,. \tag{5.89}$$

The same result was obtained in Section 2 of Chapter 4 by the stochastic process approach.

It may be of interest to compute the eigenvalues of K_9, that is, the zeros of $D_9(\lambda)$. The first six eigenvalues $\lambda_n(1)$ $(n = 1, \cdots, 6)$ are

$$\lambda_1(1) = 12.36236\,, \quad \lambda_2(1) = 485.5188\,, \quad \lambda_3(1) = 3806.546\,,$$

$$\lambda_4(1) = 14617.27\,, \quad \lambda_5(1) = 39943.83\,, \quad \lambda_6(1) = 89135.41\,.$$

Since $\mathcal{L}(U_9) = \mathcal{L}\left(\sum_{n=1}^{\infty} Z_n^2/\lambda_n(1)\right)$, where $\{Z_n\} \sim \mathrm{NID}(0,1)$, and it holds that $E(U_9) = 1/12$, only the first term $Z_1^2/\lambda_1(1)$ in the infinite sum expression for U_9 has quite a high relative weight. In fact

$$E\left(Z_1^2/\lambda_1(1)\right)/E(U_9) = \frac{12}{12.36236} = 0.9707\,.$$

This observation leads us to approximate the distribution of U_9 just by $\chi^2(1)$ divided by $\lambda_1(1)$. The both distributions have monotone densities with an infinite peak at the origin. The graphical comparison will be made in the next chapter.

The case for the two-fold integrated Brownian motion can be dealt with similarly. The statistic is

$$\begin{aligned} U_{10} &= \int_0^1 F_2^2(t)dt \\ &= \int_0^1 \int_0^1 \left[\int_{\max(s,t)}^1 \frac{1}{4}((u-s)(u-t))^2 du\right] dw(s)dw(t)\,, \end{aligned} \tag{5.90}$$

where the kernel appearing above, denoted as $K_{10}(s,t)$, has the expression

$$K_{10}(s,t) = \frac{1}{120}\,(1-t)^3(t^2 + 3t + 6 - 5st + 10s^2 - 15s)$$

for $s \le t$. We obtain

$$f^{(6)}(t) + \lambda f(t) = 0\,, \quad f(1) = f'(1) = f''(1) = f'''(0) = f^{(4)}(0) = f^{(5)}(0) = 0\,,$$

where the differential equation has the solution

$$f(t) = c_1 e^{i\alpha t} + c_2 e^{-i\alpha t} + c_3 e^{i\beta t} + c_4 e^{-i\beta t} + c_5 e^{i\gamma t} + c_6 e^{-i\gamma t}$$

for arbitrary constants c_1 through c_6 with $\alpha = \lambda^{\frac{1}{6}}$, $\beta = \lambda^{\frac{1}{6}}(1 - \sqrt{3}i)/2$ and $\gamma = \lambda^{\frac{1}{6}}(-1 - \sqrt{3}i)/2$. The six boundary conditions above yield $M(\lambda)c = 0$, where $c = (c_1,\, c_2,\, c_3,\, c_4,\, c_5,\, c_6)'$ and

$$M(\lambda) = \begin{pmatrix} e^{i\alpha} & e^{-i\alpha} & e^{i\beta} & e^{-i\beta} & e^{i\gamma} & e^{-i\gamma} \\ \alpha e^{i\alpha} & -\alpha e^{-i\alpha} & \beta e^{i\beta} & -\beta e^{-i\beta} & \gamma e^{i\gamma} & -\gamma e^{-i\gamma} \\ \alpha^2 e^{i\alpha} & \alpha^2 e^{-i\alpha} & \beta^2 e^{i\beta} & \beta^2 e^{-i\beta} & \gamma^2 e^{i\gamma} & \gamma^2 e^{-i\gamma} \\ \alpha^3 & -\alpha^3 & \beta^3 & -\beta^3 & \gamma^3 & -\gamma^3 \\ \alpha^4 & \alpha^4 & \beta^4 & \beta^4 & \gamma^4 & \gamma^4 \\ \alpha^5 & -\alpha^5 & \beta^5 & -\beta^5 & \gamma^5 & -\gamma^5 \end{pmatrix},$$

$$|M(\lambda)| = 12i\left[\cos\alpha\left(\cos\alpha + 2\cosh^2\frac{\sqrt{3}\alpha}{2} + 3\right) + 8\cos\frac{\alpha}{2}\cosh\frac{\sqrt{3}\alpha}{2} + 4\right].$$

The FD of K_{10} is found to be

$$\begin{aligned} D_{10}(\lambda) &= \frac{1}{9}\left[2\left(1+\cos\lambda^{\frac{1}{6}} + \cos\lambda^{\frac{1}{6}}\omega + \cos\lambda^{\frac{1}{6}}\omega^2\right)\right. \\ &\qquad \left. + \cos\lambda^{\frac{1}{6}}\cos\lambda^{\frac{1}{6}}\omega\cos\lambda^{\frac{1}{6}}\omega^2\right], \end{aligned} \tag{5.91}$$

where $\omega = (1+\sqrt{3}i)/2$.

The smallest eigenvalue $\lambda_1(2)$ of K_{10}, that is, the smallest zero of $D_{10}(\lambda)$ is 121.259, while $E(U_{10}) = \dfrac{1}{120}$. Thus the first term in the expression $\mathcal{L}(U_{10}) = \mathcal{L}\left(\sum_{n=1}^{\infty} Z_n^2/\lambda_n(2)\right)$ has the relative weight

$$E\left(Z_1^2/\lambda_1(2)\right)/E(U_{10}) = \frac{120}{121.259} = 0.9896\,.$$

The approximation of $\mathcal{L}(U_{10})$ by $\mathcal{L}\left(Z_1^2/\lambda_1(2)\right)$ will also be graphically presented in the next chapter.

For the general g-integrated Brownian motion the statistic takes the form as in (5.8), where the kernel has the expression

$$\begin{aligned} K(s,t) &= \frac{1}{(g!)^2}\int_{\max(s,t)}^{1}((u-s)(u-t))^g\,du \\ &= \frac{1}{(g!)^2}\sum_{j,k=0}^{g}\binom{g}{j}\binom{g}{k}\frac{(-1)^{j+k}}{j+k+1}\left[1-(\max(s,t))^{j+k+1}\right]s^{g-j}\,t^{g-k}\,. \end{aligned}$$

As a function of t, $K(s,t)$ is a polynomial of degree $2g+1$ whose coefficient is

$$\begin{aligned} \frac{(-1)^{g+1}}{(g!)^2}\sum_{k=0}^{g}\binom{g}{k}\frac{(-1)^k}{g+k+1} &= \frac{(-1)^{g+1}}{(g!)^2}\frac{g!\,\Gamma(g+1)}{\Gamma(2g+2)} \\ &= \frac{(-1)^{g+1}}{(2g+1)!}\,. \end{aligned}$$

The integral equation (5.10) with $K(s,t)$ given above yields

$$f^{(2g+2)}(t) + (-1)^g\lambda f(t) = 0\,,$$

$$f^{(j)}(1) = 0\ (j=0,1,\cdots,g)\,,\qquad f^{(k)}(0) = 0\ (k=g+1,g+2,\cdots,2g+1)\,.$$

In principle we can solve the differential equation and the $2g+2$ boundary conditions will yield a homogeneous equation to obtain the FD.

Problems

4.1 Show that the kernel appearing in (5.45) is positive definite.

4.2 Show that (5.47) is equivalent to the integral equation (5.10) with $K(s,t) = (1-(\max(s,t))^{2m+1})/(2m+1)$.

4.3 Show that the two boundary conditions in (5.47) imply $|M(\lambda)| = 0$, where $M(\lambda)$ is defined in (5.52).

4.4 Prove that $D_2(\lambda)$ in (5.57) is the FD of $K_2(s,t) = (\max(s,t))^m/2$, using the relation

$$J_\nu(z) = \frac{z}{2\nu}\{J_{\nu-1}(z) + J_{\nu+1}(z)\}\,.$$

4.5 Show that $D_2(\lambda)$ in (5.57) reduces to $1 - \dfrac{\lambda}{2}$ when $m = 0$.

4.6 Derive the c.f. of U_3 defined in (5.58).

4.7 Establish (5.62) by deriving the c.f. of the distribution on the right side.

4.8 Derive the c.f. of U_4 defined in (5.63).

4.9 Derive $\tilde{D}_5(\lambda)$ in (5.79) from the differential equation (5.74) with the boundary conditions $f(0) = f(1) = 0$ and the first condition in (5.76).

4.10 Derive the c.f. of U_7 defined in (5.84).

4.11 Derive the c.f. of U_8 defined in (5.86).

5.5. The Fredholm theory: the nonhomogeneous case

We have so far discussed how to obtain the c.f.'s of purely quadratic functionals of the Brownian motion. In this section we deal with quadratic plus linear or bilinear functionals of the Brownian motion. More specifically we consider

$$S_1 = \int_0^1 \int_0^1 K(s,t)dw(s)dw(t) + a\int_0^1 l(t)dw(t) \tag{5.92}$$

or

$$S_2 = \int_0^1 \int_0^1 K(s,t)dw(s)dw(t) + aZ\int_0^1 l(t)dw(t) + bZ^2\,, \tag{5.93}$$

where $l(t)$ is continuous, a and b are constants, while Z follows N(0,1) and is independent of $\{w(t)\}$. We continue to assume that $K(s,t)$ is continuous, symmetric and nearly definite.

To derive the c.f.'s of the above statistics we need the Fredholm theory on nonhomogeneous integral equations, which we describe briefly. Let us consider

$$f(t) = \lambda\int_0^1 K(s,t)f(s)ds + g(t)\,, \tag{5.94}$$

where $g(t)$ is a continuous function on $[0,1]$. The corresponding algebraic system is

$$f_T = \frac{\lambda}{T}K_T f_T + g_T\,, \tag{5.95}$$

where f_T and K_T are defined in (5.11), while $g_T = (\!(\, g(j/T)\,)\!)$ is a $T\times 1$ vector. Putting $D_T(\lambda) = |I_T - \lambda K_T/T|$ and assuming that $D_T(\lambda)\neq 0$, (5.95) can be solved to obtain

$$f_T = \frac{1}{D_T(\lambda)}G_T(\lambda)g_T\,,$$

where $G_T(\lambda)$ is the adjoint matrix of $I_T - \lambda K_T/T$. The j-th component of this solution may be written as

$$f\left(\frac{j}{T}\right) = \frac{1}{D_T(\lambda)}\left[G_T(j,j;\lambda)\,g\left(\frac{j}{T}\right) + \sum_{\substack{k=1\\k\neq j}}^{T} G_T(j,k;\lambda)\,g\left(\frac{k}{T}\right)\right]\,, \tag{5.96}$$

where $G_T(j,k;\lambda)$ is the (j,k)-th element of $G_T(\lambda)$. It is noted that $G_T(j,j;\lambda)$ is of the same type as $D_{T-1}(\lambda)$ so that

$$\lim_{T\to\infty} G_T(j,j;\lambda) = \lim_{T\to\infty} D_T(\lambda) = D(\lambda)\,.$$

Then we take a limit of (5.96), which can be expressed, if $D(\lambda) \neq 0$, as

$$
\begin{aligned}
f(t) &= g(t) + \int_0^1 \frac{1}{D(\lambda)} G(t,s;\lambda) g(s) ds \\
&= g(t) + \lambda \int_0^1 \Gamma(t,s;\lambda) g(s) ds \,,
\end{aligned}
\tag{5.97}
$$

where $G(s,t;\lambda)/\lambda$ is called the *Fredholm minor* and $\Gamma(s,t;\lambda) = G(s,t;\lambda)/(\lambda D(\lambda))$ the *Fredholm resolvent* or simply the resolvent of $K(s,t)$.

The above arguments are just formal, but can be made rigorous (see, for example, Courant and Hilbert (1953) and Hochstadt (1973)).

In subsequent discussions the resolvent $\Gamma(s,t;\lambda)$ plays a fundamental role, for which various expressions are possible. We have

$$
\begin{aligned}
\Gamma(s,t;\lambda) &= \sum_{j=1}^{\infty} \lambda^{j-1} K_{(j)}(s,t) \\
&= \sum_{n=1}^{\infty} \frac{1}{\lambda_n - \lambda} f_n(s) f_n(t) \\
&= K(s,t) + \lambda \int_0^1 \Gamma(s,u;\lambda) K(u,t) du \,,
\end{aligned}
\tag{5.98}
$$

where $K_{(j)}(s,t)$ is the iterated kernel defined by

$$
\begin{aligned}
K_{(j)}(s,t) &= \int_0^1 K(s,u) K_{(j-1)}(u,t) du \\
&= \sum_{n=1}^{\infty} \frac{1}{\lambda_n^j} f_n(s) f_n(t) \,,
\end{aligned}
\tag{5.99}
$$

with $K_{(1)}(s,t) = K(s,t)$, while $\{\lambda_n\}$ is a sequence of eigenvalues for the homogeneous integral equation (5.10), repeated as many times as their multiplicities, and $\{f_n(t)\}$ is an orthonormal sequence of eigenfunctions corresponding to $\{\lambda_n\}$. Note that $\Gamma(s,t;\lambda)$ is a symmetric function of s and t, and $\Gamma(s,t;0) = K(s,t)$.

It is known that the first expression in (5.98) is valid for $|\lambda| < 1/\max|K(s,t)|$. It follows from the second expression that $\Gamma(s,t;\lambda)$ is not an entire function of λ, unlike $D(\lambda)$, but is analytic except for simple poles at $\{\lambda_n\}$. Namely the resolvent is a meromorphic function of λ which possesses simple poles at the eigenvalues. The last expression may be most useful for obtaining $\Gamma(s,t;\lambda)$, although it is hard in general to obtain it for any s and t. For our purpose, however, it is not necessary, as will be shown below.

We now consider the statistic S_1 in (5.92). We first assume that $l(t) = K(0,t)$. The general case will be treated later. Thus we deal with

$$S = \int_0^1 \int_0^1 K(s,t)dw(s)dw(t) + a\int_0^1 K(0,t)dw(t)\,. \tag{5.100}$$

Using Mercer's theorem we have

$$\mathcal{L}(S) = \mathcal{L}\left(\sum_{n=1}^{\infty} \frac{1}{\lambda_n}\left(Z_n^2 + af_n(0)Z_n\right)\right)\,, \tag{5.101}$$

where $\{Z_n\} \sim \mathrm{NID}(0,1)$. It now follows (Problem 5.1) that

$$E(e^{i\theta S}) = (D(2i\theta))^{-\frac{1}{2}} \exp\left[\frac{(ia\theta)^2}{2} \sum_{n=1}^{\infty} \frac{f_n^2(0)}{\lambda_n(\lambda_n - 2i\theta)}\right]\,, \tag{5.102}$$

where $D(\lambda)$ is the FD of K. Moreover, using the definition of the resolvent $\Gamma(s,t;\lambda)$, we arrive at the following theorem (Tanaka (1990a) and Problem 5.2).

Theorem 5.8. *The c.f. $\phi(\theta)$ of S defined in (5.100) is given by*

$$\phi(\theta) = (D(2i\theta))^{-\frac{1}{2}} \exp\left[\frac{ia^2\theta}{4}\{\Gamma(0,0;2i\theta) - K(0,0)\}\right]\,,$$

where $D(\lambda)$ is the FD of K and $\Gamma(s,t;\lambda)$ is the resolvent of K.

To demonstrate how to obtain $\Gamma(0,0;\lambda)$ let us consider

$$\begin{aligned} U_1 &= \int_0^1 t^{2m}(w(t)+\kappa)^2 dt \\ &= \int_0^1\int_0^1 \frac{1}{2m+1}\left[1 - (\max(s,t))^{2m+1}\right] dw(s)dw(t) \\ &\qquad + 2\kappa \int_0^1 \frac{1-t^{2m+1}}{2m+1} dw(t) + \frac{\kappa^2}{2m+1}\,, \end{aligned} \tag{5.103}$$

where $m > -\dfrac{1}{2}$ and κ is a constant. Note that $(1-t^{2m+1})/(2m+1) = K_1(0,t)$, where $K_1(s,t) = (1-(\max(s,t))^{2m+1})/(2m+1)$. We already know the FD $D_1(\lambda)$ of K_1, which is given on the right side of (5.53). To obtain $\Gamma(0,0;\lambda)$ put $h(t) = \Gamma(0,t;\lambda)$ and use the last relation in (5.98) to get

$$h(t) = K_1(0,t) + \lambda\int_0^1 h(s)K_1(s,t)ds\,. \tag{5.104}$$

We can show (Problem 5.3) that (5.104) is equivalent to

$$h''(t) - \frac{2m}{t}h'(t) + \lambda t^{2m}h(t) = 0\,, \quad \lim_{t\to 0}\frac{h'(t)}{t^{2m}} = -1\,, \quad h(1) = 0\,. \tag{5.105}$$

The general solution is given by

$$h(t) = t^{\frac{2m+1}{2}}\left\{c_1 J_\nu\left(\frac{\sqrt{\lambda}}{m+1}\,t^{m+1}\right) + c_2 J_{-\nu}\left(\frac{\sqrt{\lambda}}{m+1}\,t^{m+1}\right)\right\}\,,$$

where $\nu = (2m+1)/(2(m+1))$. The two boundary conditions yield $M(\lambda)c = (-1, 0)'$, where $c = (c_1,\, c_2)'$ and $M(\lambda)$ is given in (5.52). Thus c_1 and c_2 can be uniquely determined as

$$c_1 = -\frac{\Gamma(\nu+1)}{2m+1}\left(\frac{\sqrt{\lambda}}{2(m+1)}\right)^{-\nu}, \quad c_2 = \frac{J_\nu\left(\frac{\sqrt{\lambda}}{m+1}\right)}{J_{-\nu}\left(\frac{\sqrt{\lambda}}{m+1}\right)}\left(\frac{\sqrt{\lambda}}{2(m+1)}\right)^{-\nu}\frac{\Gamma(\nu+1)}{2m+1}\,,$$

so that, by definition,

$$\begin{aligned}\Gamma(0,0;\lambda) &= h(0)\\ &= \frac{c_2}{\Gamma(-\nu+1)}\left(\frac{\sqrt{\lambda}}{2(m+1)}\right)^{-\nu}\\ &= \frac{\Gamma(\nu+1)}{(2m+1)\,\Gamma(-\nu+1)}\,\frac{J_\nu\left(\frac{\sqrt{\lambda}}{m+1}\right)}{J_{-\nu}\left(\frac{\sqrt{\lambda}}{m+1}\right)}\left(\frac{\sqrt{\lambda}}{2(m+1)}\right)^{-2\nu}.\end{aligned}$$

Therefore Theorem 5.8 yields

$$\begin{aligned}E\left(e^{i\theta U_1}\right) &= (D_1(2i\theta))^{-\frac{1}{2}}\exp\left\{i\kappa^2\theta\left(\Gamma(0,0;2i\theta) - \frac{1}{2m+1}\right)\right\}e^{i\kappa^2\theta/(2m+1)}\\ &= \left[\Gamma(-\nu+1)\,J_{-\nu}\left(\frac{\sqrt{2i\theta}}{m+1}\right)\Big/\left(\frac{\sqrt{2i\theta}}{2(m+1)}\right)^{-\nu}\right]^{-\frac{1}{2}}\\ &\quad\times\exp\left\{\frac{i\kappa^2\theta\,\Gamma(\nu+1)}{(2m+1)\,\Gamma(-\nu+1)}\,\frac{J_\nu\left(\frac{\sqrt{2i\theta}}{m+1}\right)}{J_{-\nu}\left(\frac{\sqrt{2i\theta}}{m+1}\right)}\left(\frac{\sqrt{2i\theta}}{2(m+1)}\right)^{-2\nu}\right\}.\end{aligned} \tag{5.106}$$

We note that, when $m = 0$, (5.106) reduces to

$$E\left(e^{i\theta U_1}\right) = (\cos\sqrt{2i\theta})^{-\frac{1}{2}}\exp\left\{i\kappa^2\theta\,\frac{\tan\sqrt{2i\theta}}{\sqrt{2i\theta}}\right\}\,,$$

which was obtained in Section 1 of Chapter 4 by the stochastic process approach. In that case U_1 reduces to $\int_0^1 X^2(t)dt$, where $\{X(t) = w(t) + \kappa\}$ follows a simple O-U process $dX(t) = dw(t)$ with $X(0) = \kappa$. As far as the O-U process is concerned, the stochastic process approach is more preferable to the present approach. In fact, if $\{X(t)\}$ is the O-U process defined by $dX(t) = -\beta X(t)dt + dw(t)$ with $X(0) = \kappa$, and if we consider

$$(5.107)\qquad \begin{aligned} U_2 &= \int_0^1 X^2(t)dt \\ &= \int_0^1 \left\{ e^{-\beta t} \int_0^t e^{\beta s} dw(s) + \kappa e^{-\beta t} \right\}^2 dt \\ &= \int_0^1 \int_0^1 \frac{e^{-\beta|s-t|} - e^{-\beta(2-s-t)}}{2\beta} dw(s)dw(t) \\ &\qquad + 2\kappa \int_0^1 \frac{e^{-\beta t} - e^{-\beta(2-t)}}{2\beta} dw(t) + \kappa^2 \frac{1 - e^{-2\beta}}{2\beta}, \end{aligned}$$

we can still follow the present approach noting that $(e^{-\beta t} - e^{-\beta(2-t)})/(2\beta) = K_2(0, t)$, where

$$K_2(s,t) = \frac{1}{2\beta} \left[e^{-\beta|s-t|} - e^{-\beta(2-s-t)} \right].$$

Since the FD of K_2 is shown to be

$$(5.108)\qquad D_2(\lambda) = \left(\cos\sqrt{\lambda - \beta^2} + \beta \frac{\sin\sqrt{\lambda - \beta^2}}{\sqrt{\lambda - \beta^2}} \right) e^{-\beta},$$

which can also be deduced from (4.9), and it is also shown that the resolvent $\Gamma(s, t; \lambda)$ of K_2 evaluated at the origin is

$$\Gamma(0,0;\lambda) = \frac{\sin\sqrt{\lambda - \beta^2}}{\sqrt{\lambda - \beta^2}} \Bigg/ \left[\cos\sqrt{\lambda - \beta^2} + \beta \frac{\sin\sqrt{\lambda - \beta^2}}{\sqrt{\lambda - \beta^2}} \right],$$

we obtain

$$(5.109)\qquad E\left(e^{i\theta U_2}\right) = \left(\cos\mu + \beta \frac{\sin\mu}{\mu} \right)^{-\frac{1}{2}} \exp\left\{ \frac{\beta}{2} + \frac{i\kappa^2\theta \frac{\sin\mu}{\mu}}{\cos\mu + \beta \frac{\sin\mu}{\mu}} \right\},$$

where $\mu = \sqrt{2i\theta - \beta^2}$. This last result, however, was obtained in (4.9) more easily by the stochastic process approach.

The present approach may be effectively used to obtain the c.f. of

$$(5.110)\qquad \begin{aligned} U_3 &= \int_0^1 t^m (w(t) + \kappa) dw(t) \\ &= \int_0^1 \int_0^1 \frac{1}{2} (\max(s,t))^m dw(s)dw(t) + 2\kappa \int_0^1 \frac{1}{2} t^m dw(t) - \frac{1}{2(m+1)}. \end{aligned}$$

It follows (Problem 5.4) that

$$E\left(e^{i\theta U_3}\right) = \left[\Gamma(\nu)J_{\nu-1}\left(\frac{\sqrt{-4i\theta m}}{m+1}\right)\bigg/\left(\frac{\sqrt{-4i\theta m}}{2(m+1)}\right)^{\nu-1}\right]^{-\frac{1}{2}} \times \exp\left\{\frac{i\kappa^2\theta}{2}\,\frac{\Gamma(-\nu+1)}{\Gamma(\nu+1)}\,\frac{J_{-\nu+1}\left(\frac{\sqrt{-4i\theta m}}{m+1}\right)}{J_{\nu-1}\left(\frac{\sqrt{-4i\theta m}}{m+1}\right)}\left(\frac{\sqrt{-4i\theta m}}{2(m+1)}\right)^{2\nu} - \frac{i\theta}{2(m+1)}\right\}, \tag{5.111}$$

where $\nu = -m/(m+1)$.

We have so far dealt with statistics that take the form as in (5.92) under the assumption that $l(t) = K(0,t)$. If this is not the case, Theorem 5.8 does not apply. For such cases Nabeya (1992) presented a solution, which we describe below. Let us consider

$$S_Y = \int_0^1 \{Y(t) + m(t)\}^2 dt\,, \tag{5.112}$$

where $\{Y(t)\}$ is a zero-mean Gaussian process with $\mathrm{Cov}(Y(s),\, Y(t)) = K(s,t)$, while $m(t)$ is a continuous function.

We also define

$$\frac{c_n}{\sqrt{\lambda_n}} = \int_0^1 m(t)f_n(t)dt\,, \quad q(t) = \sum_{n=1}^{\infty}\frac{c_n}{\sqrt{\lambda_n}}\,f_n(t)\,, \quad r(t) = m(t) - q(t)\,,$$

where $\{\lambda_n\}$ is a sequence of eigenvalues of the positive definite kernel $K(s,t)$, repeated as many times as their multiplicities, while $\{f_n(t)\}$ is an orthonormal sequence of eigenfunctions corresponding to $\{\lambda_n\}$. Note that $c_n/\sqrt{\lambda_n}$ and $q(t)$ are the Fourier coefficients and Fourier series for $m(t)$, respectively, where the infinite series is assumed to converge uniformly. It holds (Problem 5.5) that

$$\int_0^1 q(t)r(t)dt = 0 \tag{5.113}$$

so that

$$\int_0^1 m(t)q(t)dt = \int_0^1 q^2(t)dt = \sum_{n=1}^{\infty}\frac{c_n^2}{\lambda_n}\,.$$

Let $\{Z_n\}$ be a sequence of $\mathrm{NID}(0,1)$ and define

$$Z(t) = \sum_{n=1}^{\infty}\frac{f_n(t)}{\sqrt{\lambda_n}}\,Z_n\,. \tag{5.114}$$

Then $\{Z(t)\}$ is also a zero-mean Gaussian process with

$$\mathrm{Cov}(Z(s), Z(t)) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} f_n(s) f_n(t) = K(s,t)$$

so that any finite-dimensional distribution of $\{Z(t) + m(t)\}$ is the same as that of $\{Y(t) + m(t)\}$. Therefore the statistic S_Y in (5.112) has the same c.f. as

$$\begin{aligned}
(5.115) \qquad S_Z &= \int_0^1 (Z(t) + m(t))^2 dt \\
&= \int_0^1 \left\{ \sum_{n=1}^{\infty} \frac{f_n(t)}{\sqrt{\lambda_n}} (Z_n + c_n) + r(t) \right\}^2 dt \\
&= \sum_{n=1}^{\infty} \frac{1}{\lambda_n} (Z_n + c_n)^2 + \int_0^1 r^2(t) dt \,.
\end{aligned}$$

We now obtain (Nabeya (1992) and Problem 5.6)

$$(5.116) \qquad E\left(e^{i\theta S_Z}\right) = (D(2i\theta))^{-\frac{1}{2}} Q(\theta) \,,$$

where $D(\lambda)$ is the FD of K and

$$\begin{aligned}
Q(\theta) &= \exp\left\{ i\theta \int_0^1 r^2(t) dt + \sum_{n=1}^{\infty} \frac{i\theta c_n^2}{\lambda_n - 2i\theta} \right\} \\
&= \exp\left\{ i\theta \int_0^1 m^2(t) dt - 2\theta^2 \sum_{n=1}^{\infty} \frac{c_n^2}{\lambda_n(\lambda_n - 2i\theta)} \right\} .
\end{aligned}$$

Using the second relation of the resolvent in (5.98) and the fact that

$$\frac{c_n^2}{\lambda_n} = \int_0^1 \int_0^1 m(s) m(t) f_n(s) f_n(t) ds dt \,,$$

we can now establish the following theorem (Nabeya (1992)).

Theorem 5.9. *The c.f. of*

$$S_Y = \int_0^1 \{Y(t) + m(t)\}^2 dt$$

is given by

$$\phi(\theta) = (D(2i\theta))^{-\frac{1}{2}} \exp\left\{ i\theta \int_0^1 m^2(t) dt - 2\theta^2 \int_0^1 \int_0^1 \Gamma(s,t;2i\theta) m(s) m(t) ds dt \right\} ,$$

where $D(\lambda)$ is the FD of $K(s,t) = Cov(Y(s), Y(t))$ and $\Gamma(s,t;\lambda)$ is the resolvent of K.

In the above theorem we do not need to derive $\Gamma(s,t;\lambda)$, as Nabeya (1992) demonstrates. Multiplying by $m(s)$ the both sides of the last equation for $\Gamma(s,t;\lambda)$ in (5.98) and integrating with respect to s lead us to

$$h(t) = \int_0^1 K(s,t)m(s)ds + \lambda \int_0^1 K(s,t)h(s)ds \,, \tag{5.117}$$

where

$$h(t) = \int_0^1 \Gamma(s,t;\lambda)m(s)ds \,. \tag{5.118}$$

The nonhomogeneous equation (5.117) can be solved for $h(t)$ in the same way as before and it follows from (5.118) that

$$\int_0^1 \int_0^1 \Gamma(s,t;\lambda)m(s)m(t)dsdt = \int_0^1 h(t)m(t)dt \,. \tag{5.119}$$

As an example consider

$$\begin{aligned} U_4 &= \int_0^1 (w(t)+a+bt)^2 dt \\ &= \int_0^1 \int_0^1 [1-\max(s,t)]\, dw(s)dw(t) \\ &\qquad +2\int_0^1 \left(a+\frac{b}{2}-at-\frac{bt^2}{2}\right) dw(t) + a^2 + ab + \frac{b^2}{3} \,. \end{aligned} \tag{5.120}$$

Note that the statistic U_4 is not of the form given in (5.100) unless $b=0$. We need to rely on Theorem 5.9 rather than Theorem 5.8. The integral equation (5.117) with $K(s,t) = \mathrm{Cov}(w(s),w(t)) = \min(s,t)$ and $m(s) = a + bs$ is equivalent to

$$h''(t) + \lambda h(t) = -(a+bt) \,, \qquad h(0) = h'(1) = 0 \,,$$

where the general solution is given by

$$h(t) = c_1 \cos\sqrt{\lambda}\, t + c_2 \sin\sqrt{\lambda}\, t - \frac{1}{\lambda}(a+bt) \,.$$

From the boundary conditions $h(0) = h'(1) = 0$, we can determine c_1 and c_2 uniquely as

$$c_1 = \frac{a}{\lambda} \,, \qquad c_2 = \frac{a\sqrt{\lambda}\sin\sqrt{\lambda} + b}{\lambda\sqrt{\lambda}\cos\sqrt{\lambda}} \,.$$

Then Theorem 5.9 and (5.119) lead us to

$$(5.121)\qquad E\left(e^{i\theta U_4}\right) = (\cos\sqrt{2i\theta})^{-\frac{1}{2}}\exp\left[\frac{1}{2}\left\{a^2\sqrt{2i\theta}\tan\sqrt{2i\theta} -2ab\left(1-\frac{1}{\cos\sqrt{2i\theta}}\right)+b^2\left(\frac{\tan\sqrt{2i\theta}}{\sqrt{2i\theta}}-1\right)\right\}\right].$$

A completely similar argument applies to derive the c.f. of

$$\begin{aligned}(5.122)\qquad U_5 &= \int_0^1 (w(t)-tw(1)+a+bt)^2\,dt\\ &= \int_0^1\int_0^1\left[\frac{1}{3}-\max(s,t)+\frac{s^2+t^2}{2}\right]dw(s)dw(t)\\ &\quad +\int_0^1\left(a(1-2t)+b\left(\frac{1}{3}-t^2\right)\right)dw(t)+a^2+ab+\frac{1}{3}b^2\,.\end{aligned}$$

We can show (Problem 5.7) that

$$(5.123)\qquad E\left(e^{i\theta U_5}\right) = \left(\frac{\sin\sqrt{2i\theta}}{\sqrt{2i\theta}}\right)^{-\frac{1}{2}}\exp\left[\frac{1}{\cos\sqrt{2i\theta}+1}\left\{a(a+b)\sqrt{2i\theta}\sin\sqrt{2i\theta} +\frac{b^2}{2}\left(\frac{\sqrt{2i\theta}\cos\sqrt{2i\theta}\sin\sqrt{2i\theta}}{\cos\sqrt{2i\theta}-1}+\cos\sqrt{2i\theta}+1\right)\right\}\right].$$

Nabeya (1992) considers various statistics that take the form given in (5.112) arising in goodness of fit tests. Among them is the statistic

$$(5.124)\qquad U_6 = \int_0^1\left(Y(t)+\frac{a}{\pi}\sin^2\pi t\right)^2 dt\,,$$

where

$$\begin{aligned}\mathrm{Cov}(Y(s),\,Y(t)) &= \min(s,t)-st-\frac{1}{2\pi^2}\sin 2\pi s\,\sin 2\pi t\\ &= K_6(s,t)\,.\end{aligned}$$

We already know from the discussions below (5.83) that the FD $D_6(\lambda)$ of K_6 is given by

$$D_6(\lambda) = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\bigg/\left(1-\frac{\lambda}{4\pi^2}\right)\,.$$

Thus, proceeding in the same way as above, we obtain (Nabeya (1992) and Problem 5.8)

$$(5.125)\qquad E\left(e^{i\theta U_6}\right) = (D_6(2i\theta))^{-\frac{1}{2}} \exp\left[a^2\left\{\frac{i\theta}{4(2\pi^2-i\theta)} + \frac{\pi^2\sqrt{2i\theta}}{(2\pi^2-i\theta)^2}\,\frac{1-\cos\sqrt{2i\theta}}{\sin\sqrt{2i\theta}}\right\}\right].$$

We next deal with the quadratic plus bilinear functionals of the Brownian motion given in (5.93). Assuming that $l(t) = K(0,t)$ in (5.93) let us consider

$$(5.126)\qquad S = \int_0^1\int_0^1 K(s,t)dw(s)dw(t) + aZ\int_0^1 K(0,t)dw(t) + bZ^2\,,$$

where Z follows N(0,1) and is independent of $\{w(t)\}$. Since

$$(5.127)\qquad \mathcal{L}(S) = \mathcal{L}\left(\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\left(Z_n^2 + af_n(0)Z_nZ\right) + bZ^2\right),$$

it follows (Problem 5.9) that

$$(5.128)\qquad E\left(e^{i\theta S}\right) = \left[D(2i\theta)\left\{1 - 2ib\theta + a^2\theta^2\sum_{n=1}^{\infty}\frac{f_n^2(0)}{\lambda_n(\lambda_n-2i\theta)}\right\}\right]^{-\frac{1}{2}},$$

where $D(\lambda)$ is the FD of K. We now arrive at the following theorem (Tanaka (1990a)).

Theorem 5.10. *The c.f.* $\phi(\theta)$ *of* S *defined in (5.126) is given by*

$$\phi(\theta) = \left[D(2i\theta)\left\{1 - 2ib\theta - \frac{ia^2\theta}{2}\left(\Gamma(0,0;2i\theta) - K(0,0)\right)\right\}\right]^{-\frac{1}{2}},$$

where $D(\lambda)$ *is the FD of* K *and* $\Gamma(s,t;\lambda)$ *is the resolvent of* K.

It is now an easy matter to obtain the c.f. of

$$(5.129)\qquad \begin{aligned} U_7 &= \int_0^1 t^{2m}(w(t)+\kappa Z)^2dt \\ &= \int_0^1\int_0^1 \frac{1}{2m+1}\left[1-(\max(s,t))^{2m+1}\right]dw(s)dw(t) \\ &\qquad + 2\kappa Z\int_0^1\frac{1-t^{2m+1}}{2m+1}dw(t) + \frac{\kappa^2}{2m+1}Z^2\,, \end{aligned}$$

where $m > -\dfrac{1}{2}$. Theorem 5.10 and (5.106) yield

$$
\begin{aligned}
(5.130) \qquad E\left(e^{i\theta U_7}\right) &= \left[\Gamma(-\nu+1)\, J_{-\nu}\left(\frac{\sqrt{2i\theta}}{m+1}\right) \Big/ \left(\frac{\sqrt{2i\theta}}{2(m+1)}\right)^{-\nu}\right]^{-\frac{1}{2}} \\
&\quad \times \left[1 - \frac{2i\kappa^2\theta\Gamma(\nu+1)}{(2m+1)\Gamma(-\nu+1)}\, \frac{J_\nu\left(\frac{\sqrt{2i\theta}}{m+1}\right)}{J_{-\nu}\left(\frac{\sqrt{2i\theta}}{m+1}\right)} \left(\frac{\sqrt{2i\theta}}{2(m+1)}\right)^{-2\nu}\right]^{-\frac{1}{2}},
\end{aligned}
$$

where $\nu = (2m+1)/(2(m+1))$. A much simpler way of deriving (5.130) is to take the expectation of (5.106) with κ replaced by κZ by noting that $E\{\exp(i\theta Z^2)\} = (1-2i\theta)^{-\frac{1}{2}}$. This conditional argument was earlier presented in Section 1 of Chapter 4 (see arguments below (4.15)).

It is also an immediate consequence of (5.111) and the conditional argument that the c.f. of

$$
\begin{aligned}
(5.131) \qquad U_8 &= \int_0^1 t^m (w(t) + \kappa Z) dw(t) \\
&= \int_0^1 \int_0^1 \frac{1}{2}\left(\max(s,t)\right)^m dw(s) dw(t) + 2\kappa Z \int_0^1 \frac{1}{2} t^m dw(t) - \frac{1}{2(m+1)}
\end{aligned}
$$

is given by

$$
\begin{aligned}
(5.132) \qquad E\left(e^{i\theta U_8}\right) &= \left[\Gamma(\nu) J_{\nu-1}\left(\frac{\sqrt{-4i\theta m}}{m+1}\right) \Big/ \left(\frac{\sqrt{-4i\theta m}}{2(m+1)}\right)^{\nu-1}\right]^{-\frac{1}{2}} \\
&\quad \times \left[1 - i\kappa^2\theta\, \frac{\Gamma(-\nu+1)}{\Gamma(\nu+1)}\, \frac{J_{-\nu+1}\left(\frac{\sqrt{-4i\theta m}}{m+1}\right)}{J_{\nu-1}\left(\frac{\sqrt{-4i\theta m}}{m+1}\right)} \left(\frac{\sqrt{-4i\theta m}}{2(m+1)}\right)^{2\nu}\right]^{-\frac{1}{2}} \\
&\quad \times \exp\left\{-\frac{i\theta}{2(m+1)}\right\},
\end{aligned}
$$

where $\nu = -m/(m+1)$.

If we consider the stationary O-U process $\{X(t)\}$ defined by $dX(t) = -\beta X(t) dt + dw(t)$ with $X(0) = Z/\sqrt{2\beta} \sim \mathrm{N}(0, 1/(2\beta))$, we can also obtain the c.f. of

$$
\begin{aligned}
(5.133) \quad U_9 &= \int_0^1 X^2(t) dt \\
&= \int_0^1 \left\{e^{-\beta t} \int_0^t e^{\beta s} dw(s) + \kappa Z e^{-\beta t}\right\}^2 dt \\
&= \int_0^1 \int_0^1 K_2(s,t) dw(s) dw(t) + 2\kappa Z \int_0^1 K_2(0,t) dw(t) + \kappa^2 Z^2\, \frac{1-e^{-2\beta}}{2\beta},
\end{aligned}
$$

where $\kappa = 1/\sqrt{2\beta}$ and $K_2(s,t)$ is defined below (5.107). Since (5.109) is the c.f. of U_9 with κZ replaced by κ, we immediately obtain, by the conditional argument and (5.109),

$$(5.134) \qquad E\left(e^{i\theta U_9}\right) = e^{\frac{\beta}{2}} \left[\cos\mu + \left(\beta - \frac{i\theta}{\beta}\right) \frac{\sin\mu}{\mu}\right]^{-\frac{1}{2}} ,$$

where $\mu = \sqrt{2i\theta - \beta^2}$. The same result was earlier obtained in (4.14) by the stochastic process approach.

We now drop the assumption that $l(t) = K(0,t)$ in (5.93). Let us deal with

$$(5.135) \qquad S_Y = \int_0^1 \{Y(t) + m(t)Z\}^2 dt ,$$

where $\{Y(t)\}$ is a zero-mean Gaussian process with $\text{Cov}(Y(s), Y(t)) = K(s,t)$, while Z follows N(0,1) and is independent of $\{Y(t)\}$. Proceeding in the same way as before we can show (Problem 5.10) that

$$(5.136) \qquad \begin{aligned} S_Z &= \int_0^1 (Z(t) + m(t)Z)^2 dt \\ &= \sum_{n=1}^{\infty} \frac{1}{\lambda_n}(Z_n + c_n Z)^2 + Z^2 \int_0^1 r^2(t) dt \end{aligned}$$

has the same c.f. as S_Y in (5.135), where $Z(t)$ is defined in (5.114), c_n and $r(t)$ are the same as those defined below (5.112), while $\{Z_n\}$ follows NID$(0,1)$ and is independent of Z.

Then we can establish the following theorem (Problem 5.11).

Theorem 5.11. *The c.f. of S_Y in (5.135) is given by*

$$(5.137) \qquad \begin{aligned} \phi(\theta) = \Bigg[D(2i\theta) \Bigg\{1 - 2i\theta \int_0^1 m^2(t) dt \\ + 4\theta^2 \int_0^1 \int_0^1 m(s)m(t)\Gamma(s,t; 2i\theta) ds dt\Bigg\}\Bigg]^{-\frac{1}{2}} , \end{aligned}$$

where $D(\lambda)$ is the FD of $K(s,t) = Cov(Y(s), Y(t))$ and $\Gamma(s,t;\lambda)$ is the resolvent of K.

The double integral in (5.137) can also be evaluated in the same way as before. As an example let us consider

$$(5.138) \qquad U_{10} = \int_0^1 \{w(t) + (a + bt)Z\}^2 dt ,$$

which is an extended version of U_4 given in (5.120). Comparing the expressions for $\phi(\theta)$ in Theorems 5.9 and 5.11 we obtain immediately, from (5.121),

$$
\begin{aligned}
(5.139) \qquad E\left(e^{i\theta U_{10}}\right) &= (\cos\sqrt{2i\theta})^{-\frac{1}{2}}\Big[1 - a^2\sqrt{2i\theta}\tan\sqrt{2i\theta} \\
&\quad +2ab\left(1-\frac{1}{\cos\sqrt{2i\theta}}\right) - b^2\left(\frac{\tan\sqrt{2i\theta}}{\sqrt{2i\theta}}-1\right)\Big]^{-\frac{1}{2}}.
\end{aligned}
$$

Problems

5.1 Establish (5.102).

5.2 Prove Theorem 5.8 using (5.102), the second relation in (5.98) and Mercer's theorem.

5.3 Show that the nonhomogeneous integral equation (5.104) is equivalent to (5.105).

5.4 Prove that the c.f. of U_3 in (5.110) is given by (5.111).

5.5 Establish the relation in (5.113).

5.6 Show that the c.f. of S_Z in (5.115) is given by (5.116).

5.7 Show that the c.f. of U_5 in (5.122) is given by (5.123).

5.8 Show that the c.f. of U_6 in (5.124) is given by (5.125).

5.9 Derive (5.128) on the basis of (5.127).

5.10 Derive the second expression for S_Z in (5.136).

5.11 Prove Theorem 5.11.

5.6. Weak convergence of quadratic forms

In Chapter 3 we have presented a set of FCLT's and those theorems have been applied to establish weak convergence of various statistics. In doing so we normally construct a partial sum process associated with the statistic under consideration, from which we deduce weak convergence making use of the continuous mapping theorem. This is a two-stage procedure well accepted in the literature with wide applicability.

Here we dispense with constructing partial sums. Rather we deal with statistics directly without relating them to underlying partial sum processes. The statistics to be considered are quadratic forms in an increasing number of random variables. It will be seen that the limiting random variable in the sense of weak convergence is expressed by a double integral with respect to the Brownian motion, which we have dealt with in this chapter. Thus the c.f.'s of those random variables can be easily obtained by the Fredholm approach.

Following Nabeya and Tanaka (1988) let us consider

$$S_T = \frac{1}{T} \sum_{j,k=1}^{T} B_T(j,k) \varepsilon_j \varepsilon_k = \frac{1}{T} \varepsilon' B_T \, \varepsilon \,, \tag{5.140}$$

where $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_T)'$ and $B_T = (\!(B_T(j,k)\,)\!)$ is a $T \times T$ real symmetric matrix. For the time being we assume that $\{\varepsilon_j\} \sim$ i.i.d.$(0,1)$, under which we would like to discuss the weak convergence of S_T as $T \to \infty$. Note that S_T is a quadratic form in an increasing number of i.i.d. random variables.

Let us assume that there exists a symmetric, continuous and nearly definite function $K(s,t)$ $(\not\equiv 0)$ that satisfies

$$\lim_{T \to \infty} \max_{j,k} \left| B_T(j,k) - K\left(\frac{j}{T}, \frac{k}{T}\right) \right| = 0 \,. \tag{5.141}$$

This condition restricts the class of quadratic forms considered here. For example, $B_T(j,k) = \rho^{|j-k|}$ does not satisfy (5.141) so far as ρ is constant with $-1 \le \rho < 1$. The case of $\rho = 1$ is an exception, for which we find $K(s,t) \equiv 1$ that satisfies (5.141). Roughly speaking, it is necessary for (5.141) to hold that values of $B_T(j,k)$ for adjacent j's and k's are close enough to each other. It can be shown (Problem 6.1) that, if $B_T(j,k) = (1 - (\beta/T))^{|j-k|}$ with β being fixed, then $K(s,t) = e^{-\beta|s-t|}$ is a positive definite kernel that satisfies (5.141).

We note in passing that, although quadratic forms are our concern, ratio statistics can be equally treated if the denominator has a positive definite kernel. In fact, $P(U_T/V_T < x) = P(xV_T - U_T > 0)$, where U_T and V_T are quadratic forms with

$V_T > 0$. Then $xV_T - U_T$ has the same form as S_T in (5.140). The ratio statistics we shall deal with normally have the structure that the kernel associated with U_T is degenerate, while that with V_T is positive definite; hence the kernel associated with $xV_T - U_T$ is nearly definite.

Under the assumption (5.141) we now discuss the weak convergence of $S_T = \varepsilon' B_T \, \varepsilon / T$. We first note (Nabeya and Tanaka (1988) and Problem 6.2) that

$$R_T = \frac{1}{T}\varepsilon' B_T \, \varepsilon - \frac{1}{T}\sum_{j,k=1}^{T} K\left(\frac{j}{T}, \frac{k}{T}\right)\varepsilon_j \varepsilon_k \tag{5.142}$$

converges in probability to 0; hence it suffices to consider

$$\begin{aligned} S'_T &= \frac{1}{T}\sum_{j,k=1}^{T} K\left(\frac{j}{T}, \frac{k}{T}\right)\varepsilon_j \varepsilon_k \\ &= \frac{1}{T}\sum_{j,k=1}^{T}\left(\sum_{n=1}^{\infty}\frac{1}{\lambda_n} f_n\left(\frac{j}{T}\right) f_n\left(\frac{k}{T}\right)\right)\varepsilon_j \varepsilon_k \\ &= S'_{1T} + S'_{2T}, \end{aligned}$$

where $\{\lambda_n\}$ is a sequence of eigenvalues of K repeated as many times as their multiplicities and $\{f_n(t)\}$ is an orthonormal sequence of eigenfunctions corresponding to $\{\lambda_n\}$, while

$$\begin{aligned} S'_{1T} &= \sum_{n=1}^{M}\frac{1}{\lambda_n}\left(\frac{1}{\sqrt{T}}\sum_{j=1}^{T} f_n\left(\frac{j}{T}\right)\varepsilon_j\right)^2, \\ S'_{2T} &= \sum_{n=M+1}^{\infty}\frac{1}{\lambda_n}\left(\frac{1}{\sqrt{T}}\sum_{j=1}^{T} f_n\left(\frac{j}{T}\right)\varepsilon_j\right)^2. \end{aligned}$$

It is easy to see that, for M fixed,

$$\mathcal{L}\left(S'_{1T}\right) \longrightarrow \mathcal{L}\left(\sum_{n=1}^{M}\frac{1}{\lambda_n}Z_n^2\right) \qquad \text{as } T \to \infty,$$

where $\{Z_n\} \sim \mathrm{NID}(0,1)$, while, for every $\gamma > 0$ and $\delta > 0$,

$$\begin{aligned} P\left(|S'_{2T}| > \gamma\right) &< \frac{1}{\gamma}\sum_{n=M+1}^{\infty}\frac{1}{|\lambda_n|}\frac{1}{T}\sum_{j=1}^{T} f_n^2\left(\frac{j}{T}\right) \\ &< \delta, \end{aligned}$$

for all T and sufficiently large M. Then, letting $M \to \infty$, we obtain the following theorem (Nabeya and Tanaka (1988)).

Theorem 5.12. *Let $S_T = \varepsilon' B_T \varepsilon / T$ be defined as in (5.140) with $\{\varepsilon_j\} \sim$ i.i.d.$(0,1)$ and B_T satisfying (5.141). Then, as $T \to \infty$,*

$$\begin{aligned} (5.143) \qquad \mathcal{L}(S_T) \quad &\longrightarrow \quad \mathcal{L}\left(\sum_{n=1}^{\infty} \frac{1}{\lambda_n} Z_n^2\right) \\ &= \quad \mathcal{L}\left(\int_0^1 \int_0^1 K(s,t) dw(s) dw(t)\right), \end{aligned}$$

where $\{Z_n\} \sim NID(0,1)$, $\{\lambda_n\}$ is a sequence of eigenvalues of K and $\{w(t)\}$ is the one-dimensional standard Brownian motion.

This theorem tells us that the limiting distribution of $\varepsilon' B_T \varepsilon / T$ does not depend on the common distribution of ε as long as $\{\varepsilon_j\} \sim$ i.i.d.$(0,1)$. This implies that the invariance principle holds in Donsker's sense. The expressions for the limiting random variables are now quite familiar to us. The second expression in (5.143) is more important for our purpose, but this has been obtained indirectly from the first as a consequence of Mercer's theorem. This theorem ensures our intuition that

$$\begin{aligned} \mathcal{L}(S_T) \quad &\sim \quad \mathcal{L}\left(\frac{1}{T} \sum_{j,k=1}^{T} K\left(\frac{j}{T}, \frac{k}{T}\right) \varepsilon_j \varepsilon_k\right) \\ &\sim \quad \mathcal{L}\left(\sum_{j,k=1}^{T} K\left(\frac{j}{T}, \frac{k}{T}\right) \Delta w\left(\frac{j}{T}\right) \Delta w\left(\frac{k}{T}\right)\right) \\ &\longrightarrow \quad \mathcal{L}\left(\int_0^1 \int_0^1 K(s,t) dw(s) dw(t)\right), \end{aligned}$$

where $\Delta w(j/T) = w(j/T) - w((j-1)/T)$.

Some applications of Theorem 5.12 follow. Let us first consider

$$(5.144) \qquad S_{T1} = \frac{1}{T^2} y' y = \frac{1}{T^2} \varepsilon' C' C \varepsilon \,,$$

where $y = (y_1, \cdots, y_T)'$ with $y_j = y_{j-1} + \varepsilon_j$, $y_0 = 0$ and $\{\varepsilon_j\} \sim$ i.i.d.$(0,1)$, while C is the random walk generating matrix defined in (1.3). The statistic was already discussed in Chapters 1 and 3 by the eigenvalue and stochastic process approaches.

Noting that the (j,k)-th element of $B_T = C'C/T$ is $(T+1-\max(j,k))/T$, we can readily find $K(s,t) = 1 - \max(s,t)$ that satisfies (5.141). It is much easier to work with Theorem 5.12 to establish the weak convergence of S_{T1}.

We next consider

$$(5.145) \qquad S_{T2} = \frac{1}{T^2} y'\Omega^{-2} y \,,$$

where $y_j = \varepsilon_j - \varepsilon_{j-1}$ with $\varepsilon_0, \varepsilon_1, \cdots, \sim$ i.i.d.$(0,1)$ and $\Omega = V(y)$ defined in Section 2 of Chapter 1. Since

$$\begin{aligned} \mathcal{L}(S_{T2}) &= \mathcal{L}\left(\frac{1}{T^2}\varepsilon'\Omega^{-1}\varepsilon\right) \\ &= \mathcal{L}\left(\frac{1}{T^2}\varepsilon'\left(CC' - \frac{1}{T+1}Cee'C'\right)\varepsilon\right) \,, \end{aligned}$$

where $e = (1, \cdots, 1)'$, and the (j,k)-th element of $B_T = (CC' - Cee'C'/(T+1))/T$ is $(\min(j,k) - jk/(T+1))/T$, we find $K(s,t) = \min(s,t) - st$ that satisfies (5.141).

An extended version of the statistic S_{T1} in (5.144) is

$$(5.146) \qquad S_{T3} = \frac{1}{T^2} y'y = \frac{1}{T^2}\varepsilon' C'(\rho)C(\rho)\varepsilon \,,$$

where $y_j = \rho y_{j-1} + \varepsilon_j$ with $\rho = 1 - (\beta/T)$, $y_0 = 0$ and $\{\varepsilon_j\} \sim$ i.i.d.$(0,1)$, while

$$(5.147) \qquad C(\rho) = \begin{pmatrix} 1 & & & & & 0 \\ \rho & 1 & & & & \\ \cdot & \cdot & \cdot & & & \\ \cdot & & \cdot & \cdot & & \\ \cdot & & & \cdot & \cdot & \\ \rho^{T-1} & \cdot & \cdot & \cdot & \rho & 1 \end{pmatrix} .$$

The (j,k)-th element of $B_T = C'(\rho)C(\rho)/T$ is $(\rho^{|j-k|} - \rho^{2T-j-k+2})/(T(1-\rho^2))$ and we can arrive at (Problem 6.3)

$$\begin{aligned} (5.148) \qquad \mathcal{L}(S_{T3}) &\longrightarrow \mathcal{L}\left(\int_0^1\int_0^1 \frac{e^{-\beta|s-t|} - e^{-\beta(2-s-t)}}{2\beta}\, dw(s)dw(t)\right) \\ &= \mathcal{L}\left(\int_0^1 X^2(t)dt\right) \,, \end{aligned}$$

where $\{X(t)\}$ is the O-U process defined by $dX(t) = -\beta X(t)dt + dw(t)$ with $X(0) = 0$. The equivalence of the above two distributions were established in (5.2) and the limiting c.f. is given by $(D(2i\theta))^{-\frac{1}{2}}$, where $D(\lambda)$ is defined by the right side of (5.108).

We also consider an extended version of S_{T2} in (5.145). Let us put

$$S_{T4} = \frac{1}{T^2} y'\Omega^{-2} y + \frac{\gamma}{T^4} y'\Omega^{-3} y \,, \tag{5.149}$$

where $\{y_j\}$ follows an MA(1) process, as in (5.145). Since

$$\mathcal{L}(S_{T4}) = \mathcal{L}\left(\frac{1}{T^2}\varepsilon'\left(\Omega^{-1} + \frac{\gamma}{T^2}\Omega^{-2}\right)\varepsilon\right) \,,$$

we shall have (Tanaka (1990b) and Problem 6.4)

$$\begin{aligned} \mathcal{L}(S_{T4}) \quad &\longrightarrow \quad \mathcal{L}\left(\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n} + \frac{\gamma}{\lambda_n^2}\right) Z_n^2\right) \\ &= \quad \mathcal{L}\left(\int_0^1\int_0^1 (K(s,t) + \gamma K_{(2)}(s,t)) dw(s) dw(t)\right) \,, \end{aligned} \tag{5.150}$$

where $K(s,t) = \min(s,t) - st$ and the iterated kernel $K_{(2)}(s,t)$ is defined by

$$\begin{aligned} K_{(2)}(s,t) \quad &= \quad \int_0^1 K(s,u)K(u,t)du \\ &= \quad \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} f_n(s) f_n(t) \,. \end{aligned}$$

The c.f. of the limiting distribution in (5.150) can also be derived. In a general setting we have the following theorem (Nabeya (1989), Tanaka (1990b) and Problem 6.5).

Theorem 5.13. *Suppose that the statistic S_T is defined by*

$$S_T = \frac{1}{T}\varepsilon' B_T \varepsilon + \frac{\gamma}{T^2}\varepsilon' B_T^2 \varepsilon \,,$$

where $\{\varepsilon_j\} \sim$ i.i.d.$(0,1)$ and B_T satisfies (5.141). Then

$$\begin{aligned} \mathcal{L}(S_T) \quad &\longrightarrow \quad \mathcal{L}\left(\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n} + \frac{\gamma}{\lambda_n^2}\right) Z_n^2\right) \\ &= \quad \mathcal{L}\left(\int_0^1\int_0^1 (K(s,t) + \gamma K_{(2)}(s,t)) dw(s) dw(t)\right) \,, \end{aligned} \tag{5.151}$$

where $\{Z_n\} \sim$ NID$(0,1)$ and $\{\lambda_n\}$ is a sequence of eigenvalues of K repeated as many times as their multiplicities. Moreover the c.f. of the limiting distribution is given by

$$\lim_{T\to\infty} E\left(e^{i\theta S_T}\right) \quad = \quad \prod_{n=1}^{\infty}\left[1 - 2i\theta\left(\frac{1}{\lambda_n} + \frac{\gamma}{\lambda_n^2}\right)\right]^{-\frac{1}{2}} \tag{5.152}$$

$$= \prod_{n=1}^{\infty}\left[1-\frac{1}{\lambda_n}\left(i\theta+\sqrt{-\theta^2+2i\gamma\theta}\right)\right]^{-\frac{1}{2}}$$

$$\times\left[1-\frac{1}{\lambda_n}\left(i\theta-\sqrt{-\theta^2+2i\gamma\theta}\right)\right]^{-\frac{1}{2}}$$

$$= \left[D\left(i\theta+\sqrt{-\theta^2+2i\gamma\theta}\right)D\left(i\theta-\sqrt{-\theta^2+2i\gamma\theta}\right)\right]^{-\frac{1}{2}},$$

where $D(\lambda)$ *is the FD of* K.

This theorem is useful for computing the limiting local power of test statistics for testing an MA unit root. In fact the statistic S_{T4} in (5.149) is essentially the LM statistic evaluated under the local alternative $H_1 : \alpha = 1 - (\sqrt{\gamma}/T)$ for testing $H_0 : \alpha = 1$ against H_1 in the MA(1) model $y_j = \varepsilon_j - \alpha\varepsilon_{j-1}$. The limiting local power of the test is then easily calculated by inverting the c.f. given in (5.152). We shall discuss more details in Chapter 10 extending the model (see also Chapter 8).

Here we extend the present discussion into two directions. Suppose first that we deal with a vector-valued error process $\{\underset{\sim}{\varepsilon}_j\}$ and our concern is a statistic

$$(5.153)\qquad S_T^{(q)} = \frac{1}{T}\sum_{j,k=1}^{T} B_T(j,k)\underset{\sim}{\varepsilon}_j' H \underset{\sim}{\varepsilon}_k = \frac{1}{T}\varepsilon'(B_T \otimes H)\varepsilon\,,$$

where H is a nonzero $q\times q$ symmetric matrix with constant elements, $\{\underset{\sim}{\varepsilon}_j\} \sim$ i.i.d.$(0, I_q)$ and $\varepsilon = (\underset{\sim}{\varepsilon}_1', \cdots, \underset{\sim}{\varepsilon}_T')'$, while $B_T(j,k)$ is the same as before and has a uniform limit $K(s,t)$ as in (5.141) which is symmetric, continuous and nearly definite. It is now an easy matter to establish (Problem 6.6) that

$$(5.154)\qquad \mathcal{L}\left(S_T^{(q)}\right) \longrightarrow \mathcal{L}\left(\int_0^1\int_0^1 K(s,t)dw'(s)Hdw(t)\right),$$

where $\{w(t)\}$ is the q-dimensional standard Brownian motion. The expression on the right side was discussed in Chapter 2 and it holds (Problem 6.7) that

$$(5.155)\qquad \lim_{T\to\infty} E\left(e^{i\theta S_T^{(q)}}\right) = \prod_{j=1}^{q}(D(2i\delta_j\theta))^{-\frac{1}{2}},$$

where $D(\lambda)$ is the FD of K while δ_j's are the eigenvalues of H.

As an application of the above result let us consider the model:

$$(5.156)\qquad y_j = \rho y_{j-m} + \varepsilon_j\,,\qquad (j = 1, \cdots, T)\,,$$

where $y_{1-m} = y_{2-m} = \cdots = y_0 = 0$ and $\{\varepsilon_j\} \sim$ i.i.d.$(0,1)$. The model may be referred to as a seasonal AR model with period m. Suppose that m is a divisor of T so that $N = T/m$ is an integer. Then we have

$$y = (C(\rho) \otimes I_m)\varepsilon\,,$$

where $y = (y_1, \cdots, y_T)'$, $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_T)'$ and $C(\rho)$ is an $N \times N$ matrix defined in (5.147) with T replaced by N. Assuming that $\rho = 1 - (\beta/N)$ with β fixed we can show (Problem 6.8) that, as $N \to \infty$,

$$\mathcal{L}\left(\frac{1}{N^2}\, y'y\right) \longrightarrow \mathcal{L}\left(\int_0^1 \int_0^1 \frac{e^{-\beta|s-t|} - e^{-\beta(2-s-t)}}{2\beta}\, dw'(s)dw(t)\right)\,, \tag{5.157}$$

where $\{w(t)\}$ is the m-dimensional standard Brownian motion. Thus we have, from (5.108) and (5.155),

$$\lim_{N\to\infty} E\left[\exp\left(\frac{i\theta}{N^2}\, y'y\right)\right] = \left(\cos\sqrt{2i\theta - \beta^2} + \beta\,\frac{\sin\sqrt{2i\theta-\beta^2}}{\sqrt{2i\theta-\beta^2}}\right)^{-\frac{m}{2}} e^{\beta m/2}\,.$$

The assumption that m is a divisor of T is not a restriction so far as the asymptotic result is concerned.

Another extension of the class of quadratic forms is to relax the i.i.d. assumption on the error process $\{\varepsilon_j\}$. For this purpose we consider

$$V_T = \frac{1}{T}\sum_{j,k=1}^{T} B_T(j,k)u_j u_k = \frac{1}{T}\, u' B_T\, u\,, \tag{5.158}$$

where we assume that $\{u_j\}$ is generated by

$$u_j = \sum_{l=0}^{\infty} \alpha_l\, \varepsilon_{j-l}\,, \qquad \sum_{l=0}^{\infty} |\alpha_l| < \infty\,, \qquad \alpha \equiv \sum_{l=0}^{\infty} \alpha_l \neq 0\,, \tag{5.159}$$

and $\{\varepsilon_j\} \sim$ i.i.d.$(0,1)$. Note that $\{u_j\}$ is a stationary process with a slightly weaker condition than that imposed in Section 5 of Chapter 3.

We can show (Tanaka (1990a) and Problem 6.9) that

$$R_T = V_T - \frac{1}{T}\sum_{j,k=1}^{T} K\left(\frac{j}{T}, \frac{k}{T}\right) u_j u_k \tag{5.160}$$

converges in probability to 0. Concentrating on the second term on the right side of (5.160) we can deduce (Problem 6.10) that

$$
\begin{aligned}
(5.161)\qquad \mathcal{L}(V_T) &= \mathcal{L}\left(\frac{1}{T}\sum_{j,k=1}^{T} B_T(j,k)u_j u_k\right) \\
&\longrightarrow \mathcal{L}\left(\alpha^2\int_0^1\int_0^1 K(s,t)dw(s)dw(t)\right) .
\end{aligned}
$$

The existence of a factor α^2 is a consequence of the linear process assumption in (5.159), which we also discussed in Section 5 of Chapter 3.

A more general assumption on $\{u_j\}$ in (5.158) is possible (Tanaka (1990a)), to the extent that the weak convergence result in (5.161) holds. We do not pursue the matter here.

This section has discussed weak convergence of quadratic forms, but the discussion can be extended to the class of quadratic plus linear or bilinear forms like

$$
(5.162)\qquad S_{1T} = \frac{1}{T}\sum_{j,k=1}^{T} B_T(j,k)\varepsilon_j\varepsilon_k + \frac{a}{\sqrt{T}}\sum_{k=1}^{T} c_T(k)\varepsilon_k
$$

or

$$
(5.163)\qquad S_{2T} = \frac{1}{T}\sum_{j,k=1}^{T} B_T(j,k)\varepsilon_j\varepsilon_k + \frac{a}{\sqrt{T}}Z\sum_{k=1}^{T} c_T(k)\varepsilon_k + bZ^2 ,
$$

where Z follows N(0,1) and is independent of $\{\varepsilon_j\}$. Under the assumption (5.141) on $B_T(j,k)$ and a similar assumption on $c_T(k)$, the statistics S_{1T} and S_{2T} will converge in distribution to those random variables given in (5.92) and (5.93), respectively. Then the limiting c.f.'s may be derived by the Fredholm approach.

In connection with time series problems, however, the linear or bilinear part in the above statistics arises as a result of nonnegligible influence of the initial value of an underlying process. As an example let us consider

$$
(5.164)\qquad y_j = y_{j-1} + \varepsilon_j , \qquad (j = 1, \cdots, T) .
$$

If $y_0 = \sqrt{T}\gamma$, then it holds that

$$
(5.165)\qquad \frac{1}{T^2}\sum_{j=1}^{T} y_j^2 = \frac{1}{T}\sum_{j,k=1}^{T}\frac{T+1-\max(j,k)}{T}\varepsilon_j\varepsilon_k + \frac{2\gamma}{\sqrt{T}}\sum_{k=1}^{T}\frac{T-k+1}{T}\varepsilon_k + \gamma^2 ,
$$

which has essentially the same form as S_{1T} in (5.162). We could use the Fredholm approach to derive the c.f. of the limiting distribution. A more general statistic is considered in (5.107). As was discussed there, we can take a much simpler route. We already know from Section 9 of Chapter 3 that

$$\mathcal{L}\left(\frac{1}{T^2}\sum_{j=1}^{T} y_j^2\right) \longrightarrow \mathcal{L}\left(\int_0^1 X^2(t)dt\right), \tag{5.166}$$

where $dX(t) = dw(t)$ with $X(0) = \gamma$ and the c.f. of the limiting distribution is available in (4.9) with $\alpha = 0$. Similarly, if $y_0 = \sqrt{T}\gamma Z$, $\sum\limits_{j=1}^{T} y_j^2/T^2$ has the same form as the right side of (5.165) with γ replaced by γZ. That form is the same as S_{2T} in (5.163). Then we have the same weak convergence result as in (5.166) with $X(0)$ replaced by $X(0) = \gamma Z$ and the limiting c.f. is readily obtained by the stochastic process approach.

The initial value problem associated with a simple integrated model in (5.164) or more generally a near integrated model can be solved by relating such discrete-time models to the O-U process. Thus we do not discuss more on the weak convergence of the statistics like S_{1T} in (5.162) or S_{2T} in (5.163). We shall return to the initial value problem in Chapter 7.

Problems

6.1 Show that $B_T(j,k) = (1-(\beta/T))^{|j-k|}$ with β being fixed satisfies (5.141) with $K(s,t) = e^{-\beta|s-t|}$.

6.2 Prove that R_T defined in (5.142) converges in probability to 0.

6.3 Establish the weak convergence result in (5.148).

6.4 Establish the weak convergence result in (5.150).

6.5 Prove Theorem 5.13.

6.6 Deduce the weak convergence result in (5.154).

6.7 Show that the limiting c.f. of $S_T^{(q)}$ in (5.154) is given by (5.155).

6.8 Establish the weak convergence result in (5.157).

6.9 Prove that R_T defined in (5.160) converges in probability to 0.

6.10 Deduce the weak convergence result in (5.161).